Solution of differential equation of motion of Coil of Ballistic Galvanometer

Differential equation of motion of Coil of B.G. is

$$\frac{d^2\theta}{dt^2} + 2b\frac{d\theta}{dt} + k^2\theta = f \longrightarrow (1)$$

where $b = \frac{b_1}{2I_m}$, $b_1 = a_1 + \frac{G_0^2}{R}$, $k = \sqrt{\frac{C}{I_m}}$

$$f = \frac{G_0 I}{R I_m}$$

The general solution of equation (1) is the sum of the complementry function and particular integral. The Complementry function is the solution of equation

$$\frac{d^2\theta}{dt^2} + 2b\frac{d\theta}{dt} + k^2\theta = 0 \longrightarrow (2)$$

Let the solution of this equation is

$$\theta = Ae^{\alpha t}$$

$$\therefore \frac{d\theta}{dt} = A\alpha e^{\alpha t}$$

$$\therefore \frac{d^2\theta}{dt^2} = A\alpha^2 e^{\alpha t}$$

$$\therefore (2) \Rightarrow A\alpha^2 e^{\alpha t} + 2bA\alpha e^{\alpha t} + k^2 A e^{\alpha t} = 0$$

$$\Rightarrow Ae^{\alpha t}(\alpha^2 + 2b\alpha + k^2) = 0$$

$$\therefore \alpha = \frac{-2b \pm \sqrt{4b^2 - 4k^2}}{2}$$

$$= -b \pm \sqrt{b^2 - k^2}$$

$\therefore \alpha = -b \pm r$ where $r = \sqrt{b^2 - k^2}$

$$\therefore \theta = A_1 e^{(-b+r)t} + A_2 e^{(-b-r)t} \longrightarrow (3)$$

where A_1 and A_2 two constants.

The particular integral will be $\theta = \frac{f}{k^2}$ because it satisfied the equation (1) by substitution of $\theta = \frac{f}{k^2}$ on it.

Hence total solution becomes -

$$\theta = e^{-bt}\left[A_1 e^{rt} + A_2 e^{-rt}\right] + \frac{f}{k^2} \longrightarrow (4)$$

If the galvanometer is used to measure steady current, The final deflection θ_0 is obtained when time t is large ($t \to \infty$) and equation (4) becomes

$$\theta_0 = \frac{f}{k^2}$$

To find out the constant A_1 and A_2, we consider the condition when $t = 0$, $\theta = 0$, $\frac{d\theta}{dt} = 0$,

$$\therefore (4) \Rightarrow 0 = A_1 + A_2 + \theta_0 \longrightarrow (5)$$

$$(4) \Rightarrow \frac{d\theta}{dt} = 0 = -b(A_1 + A_2) + r(A_1 - A_2)$$

$$\Rightarrow -b(A_1 + A_2) + r(A_1 - A_2)$$

$$\Rightarrow A_1(r-b) - A_2(r+b) = 0 \rightarrow (6)$$

Now

$$(5) \times (r+b) \Rightarrow A_1(r+b) + A_2(r+b) = -\theta_0(r+b) \longrightarrow (7)$$

⑥ + ⑦ $\Rightarrow A_1(2\gamma) = -\theta_0(\gamma+b)$

$$\therefore A_1 = -\frac{(\gamma+b)\theta_0}{2\gamma}$$

⑤ $\times (\gamma-b) \Rightarrow A_1(\gamma-b) + A_2(\gamma-b) = -\theta_0(\gamma-b) \rightarrow$ ⑧

⑥ − ⑧ $\Rightarrow -A_2(2\gamma) = \theta_0(\gamma-b)$

$$\therefore A_2 = -\frac{(\gamma-b)\theta_0}{2\gamma}$$

$$\therefore ④ \Rightarrow \theta = -e^{-bt}\left[\frac{\gamma+b}{+2\gamma} e^{\gamma t} + \frac{\gamma-b}{2\gamma} e^{-\gamma t}\right]\theta_0 + \theta_0 \rightarrow ⑨$$

where $\gamma = \sqrt{b^2-k^2}$

$$\therefore \theta = \theta_0\left[1 - e^{-bt}\left(\frac{\gamma+b}{2\gamma} e^{\gamma t} - \frac{\gamma-b}{2\gamma} e^{-\gamma t}\right)\right] \rightarrow ⑩$$

There are three possible cases

Case ① If 'r' is real i.e $b^2 > k^2$, equation ① indicates that galvanometer deflection θ increases gradually from 0 to steady value θ_0 and the motion of coil will be non oscillatory or dead beat. The motion of coil of galvanometer is ~~known as~~ overdamped.

Case ② When $\gamma = 0$ i.e $b^2 = k^2$, the galvanometer coil deflection returns to rest. The motion of the coil is non oscillatory ~~and~~ and coil of galvanometer is critically damped.

In this case $\left[\frac{1}{2I_m}\left(a_1 + \frac{G_0}{R}\right)\right]^2 = \frac{C}{I_m}$

Case 3 If $b^2 < k^2$ or 'r' is imaginary then

$$r = (b^2 - k^2)^{1/2}$$
$$= j\sqrt{k^2 - b^2}$$
$$= jp$$

$\therefore$ (4) $\Rightarrow$ $\theta = e^{-bt}\left[A_1 e^{jpt} + A_2 e^{-jpt}\right] + \theta_0$

$e^{jpt} = \cos pt + j\sin pt$

$e^{-jpt} = \cos pt - j\sin pt$

$$= e^{-bt}\left[(A_1 + A_2)\cos pt + j(A_1 - A_2)\sin pt\right] + \theta_0$$

$$= e^{-bt}\left(A'\cos pt + B'\sin pt\right) + \theta_0$$

where $A' = A_1 + A_2$ and $B' = j(A_1 - A_2)$

$$\therefore \theta = e^{-bt}\left[C\sin\phi\cos pt + C\cos\phi\sin pt\right] + \theta_0$$

$\therefore$ $A' = C\sin\phi$

$B' = C\cos\phi$

$\therefore \phi = \tan^{-1}\frac{A'}{B'}$

$= \tan^{-1}\frac{(A_1 + A_2)}{j(A_1 - A_2)}$

$$= Ce^{-bt}\sin(pt + \phi) + \theta_0$$

This shows that the motion of the coil is simple harmonic having time period

$$T = \frac{2\pi}{p} = \frac{2\pi}{\sqrt{k^2 - b^2}}$$
$$= 2\pi\left[\frac{C}{I_m} - \frac{\left(a_1 + \frac{G_0}{R}\right)^2}{4I_m}\right]^{-1/2}$$

If galvanometer is used for charge measurement, there is no continuous source of e.m.f i.e $E = 0$ thus $\theta_0 = 0$,

$$\therefore \theta = Ce^{-bt}\sin\left(\sqrt{k^2 - b^2}\,t + \phi\right)$$

Thus the motion of galvanometer coil is under damped oscillation and the galvanometer is known as oscillatory or ballistic. Thus for ballistic motion $b^2 < k^2$ i.e $\frac{1}{2I_m}\left[a_1 + \frac{G_0}{b}\right] < \sqrt{\frac{C}{I_m}}$.

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