

Laplace's Equation and Poisson's Equation:

Gauss's law in the differential form is given by,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\text{We know that } \vec{E} = -\nabla V \quad \text{--- (1)}$$

\therefore Divergence of eqn (1) \Rightarrow (1)

$$\nabla \cdot \vec{E} = \nabla \cdot (-\nabla V)$$

$$\Rightarrow \nabla \cdot \vec{E} = -\nabla^2 V$$

$$\Rightarrow -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \quad \text{--- (2)}$$

Eqn (2) is known as Poisson's equation.

In regions where there is no charge, i.e. $\rho = 0$, Poisson's eqn reduces to Laplace's equation,

$$\boxed{\nabla^2 V = 0} \quad \text{--- (3)}$$

In cartesian coordinates Laplace's equation is given by -

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In spherical polar coordinates, we have

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0.$$

In cylindrical polar coordinates, we have,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

The Uniqueness Theorem :

The Laplace's equation is given by,

$$\boxed{\nabla^2 V = 0} \quad \text{--- (1)}$$

which is a second order differential equation.
Thus, eqn (1) have no local maxima and minima, therefore, have no equilibrium point.

We use the Laplace's equation to determine V . If potential V can be found we can get \vec{E} from it. In order to determine V , suitable boundary conditions must be applied.

Let us consider a closed boundary, having potential V_0 at the boundary. Let V_1 and V_2 be the solutions of the Laplace's eqn and satisfy the boundary condition.

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0$$

$$\text{Let } \phi(\vec{r}) = V_1 - V_2$$

$$\begin{aligned} \text{then, } \nabla^2 \phi &= \nabla^2 V_1 - \nabla^2 V_2 \\ &= 0 \end{aligned}$$

This obeys Laplace's equation and it takes the value zero on all boundaries (since both V_1 and V_2 satisfies the same boundary condition) which implies that there can be only one solution of Laplace's

equation that satisfies the prescribed boundary conditions. This is the Uniqueness theorem.