

Plasma State

Three states of matter, known as solid, liquid and gas, are generally known to a common person. Any one of these states can, in general, be converted into another through the exchange of energy. In day-to-day life, H_2O molecule is a remarkable example, as it is found in all the three states, as ice (solid), water (liquid) and steam (gas). Obviously, by supplying energy to the matter, solid state can be converted into a liquid one, and a liquid state into a gaseous one; for the reverse process, the energy is extracted out from the matter. In some cases, it is possible to convert solid state into a gaseous state. NH_4Cl is example of such case. Further supply of energy to a matter in the gaseous state, breaks the molecules into its constituent atoms, and then the atoms are stripped off their electrons producing, positive ions and negatively charged electrons. The amount of energy required to liberate an electron from an atom is known as the *ionization potential*¹. This supplied energy may be in the form of heat, radiation, due to collision or due to chemical reaction. Ionization due to heat occurs at very high temperature of the order of million Kelvin which can be produced in the laboratories. This state of matter where charged as well as neutral particles exist simultaneously is generally known as plasma. This term 'plasma' was first coined by Langmuir in 1923. Thus, the plasma may be defined as the following:

~~Def~~ ✓ *Plasma is a quasi-neutral gas of charged and neutral particles which exhibit collective behaviour.*

As the plasma is found in a natural form in a number of cosmic objects and in the upper atmosphere of the earth, therefore, it is sometimes

¹From an atom, having more than one electron, electrons may be liberated one by one. Thus, an atoms can have several ionization potentials; each subsequent ionization of an atom has larger value of ionization potential.

defined as

the fourth state of matter.

When the gas is ionized, its dynamical behaviour is influenced by the external electric and magnetic fields. Moreover, the separated charged particles within the plasma give rise to new forces between the constituent particles. Thus, the properties of the plasma become quite different from those of neutral atoms and molecules. In this chapter we shall discuss some processes and properties of plasma.

1.1 Natural plasma

Natural plasma exists in some cosmic objects like interiors and atmospheres of hot stars, planetary nebulae, regions of ionized hydrogen in the interstellar medium, and the upper atmosphere of the earth. On the earth, plasma can however be produced in the laboratories. The reason for finding natural plasma in the cosmic objects and not at the earth is due to the large differences in density and temperature in the two regions. On the earth, the density is very large and temperature is very low as compared to those in the cosmic objects. Some regions where natural plasma is found are as the following.

1.1.1 Ionosphere

Streams of charged particles, known as *solar wind*, are being continuously emitted by our sun. Some of these charged particles reach up to the upper atmosphere of the earth. Moreover, intense radiations (γ -rays, x-rays, UV radiations) coming from the outer space ionize the upper atmosphere of the earth. This upper atmosphere of the earth is known as the *ionosphere*. It is about 50 km above the earth's surface. Since hazardous radiations (γ -rays, x-rays, UV radiations) are absorbed in the upper atmosphere of the earth, the earth's atmosphere thus plays important role in our life by shielding us from the hazardous radiations coming from the outer space. This ionosphere is used for communication purposes, as the radiations of frequency less than the plasma frequency are reflected back by the plasma in the ionosphere .

1.1.2 van Allen belts

The investigations made by satellites found two regions, known as the *van Allen radiation belts*, which envelop the earth. One belt is at a distance of about 9700 km and second at 22,500 km from the surface of the earth. The thickness of inner and outer belts is respectively about 4800 km and 8000 km. The belts contain charged particles (ions) trapped between the magnetic lines of force.

1.1.3 Aurorae

Above the earth's magnetic poles, the charged particles have free access to the earth's surface, as the lines of magnetic field are concentrated towards the surface. These charged particles interact with the molecules of the upper air, causing a glow from time to time. These glow are the aurorae which are also known as the northern lights and southern lights.

1.1.4 Solar corona

Two strong emission lines at λ 5303 Å and 6374 Å were found in the solar atmosphere. Later on some more lines were found in the spectra of the sun. These lines could not be assigned to any of the known atoms or their singly ionized ions. Since the estimated temperature of the photosphere around the sun is about 6000 K, scientists could expect either atoms or their singly ionized ions in the solar atmosphere. In absence of any atom to which these lines could be assigned, scientist coined a name coronium to some unknown atom. No one knew about the physical properties of this coronium, except to say that it was responsible for generation of those unassigned lines. Later on, through laboratory studies at very high temperatures it was found that these unknown lines could be generated by highly ionized ions. For example, the lines at λ 5303 Å and 6374 Å are generated by Fe XIV (thirteen times ionized iron, Fe^{+13}) and Fe X (nine times ionized iron, Fe^{+9}), respectively. Since these highly ionized ions are produced at a temperature of million Kelvin, their was no alternative but to accept that the temperature in the corona around the sun is of million Kelvin.

In spite of energy losses due to radiation as well as conduction, the temperature of corona is found maintained at million Kelvin. It is indeed

a challenging task before the scientists to find out the source for the solar coronal heating. Such coronae are found around a number of stars.

1.1.5 Core of the sun

As an explanation of the source of energy in the sun, it is now well established that in the core, the temperature is of the order of 15 million K and the process of fusion of four hydrogen nuclei into a helium nucleus is going on. Thus, the core of the sun is so hot that the matter there is in the plasma state. Besides the fusion of hydrogen nuclei in a helium nucleus, in some stars, CNO cycle or triple α reactions are going on. Consequently, in the cores of the shining stars, the temperature is sufficiently high to maintain the nuclear reactions. Hence, the material in the cores of stars is in the plasma state.

1.1.6 HII regions

In some parts of the interstellar medium, temperature is so high that the hydrogen gas is in the ionized state. These regions are generally known as the HII regions. These regions are either associated with the evolved stars (stars in the late stage) or they form a big cloud in which the process of star formation is going on.

Other alternative expressions for electron temperature have been derived by other scientists. However, T_e is found proportional to (E/P) up to 1.5 V/cm mm Hg in argon and neon.

1.3. Debye shielding

A fundamental characteristic of plasma is that it can shield out electric potentials that are applied to it. In order to understand it, let us put an electric field inside a plasma by inserting two charged balls connected to a battery (Figure 1.2). Here, we assume that a layer of dielectric restricts the plasma from recombining on the surface or the battery is large enough to maintain the potential in spite of recombination. Now, the balls almost immediately would attract particles of opposite charge and a cloud of positive ions would be formed around the negative ball and a cloud of electrons would be formed around the positive ball.

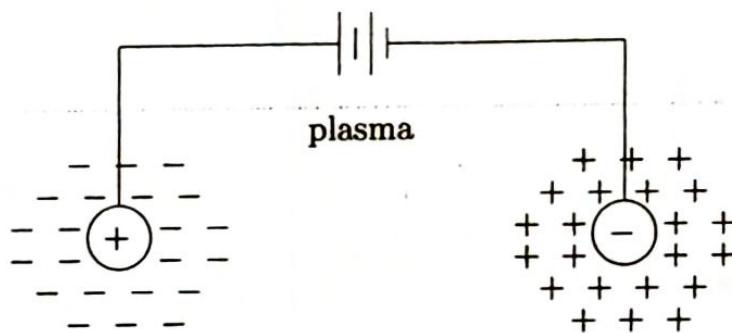


Figure 1.2: Shows two charged balls connected to a battery and inserted inside a plasma

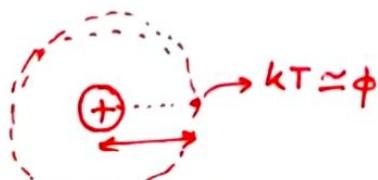
In case the plasma is $T \sim 0$ cold and there are no thermal motions of charged particles, the surrounding cloud in each case would have as many charges as there are in the ball. Thus, in the plasma, outside the clouds, there would be no electric field and the shielding is perfect. On the other hand, if the plasma temperature is finite, the particles at the edge of the cloud (there electric field is weak), have enough thermal energy to escape from the electrostatic potential well. The edge of the cloud is estimated to occur at the radial distance where the potential energy is approximately equal to the thermal kinetic energy KT of the particles. Consequently, the shielding is not complete.

$$\frac{1}{2}mv^2 = \frac{3}{2}kT$$

$$T \sim 0$$

$$v \sim 0$$

$$\phi \approx KT$$



Let us calculate the approximate thickness of such a charge cloud. Suppose the potential ϕ on the plane $x = 0$ is kept at a value ϕ_0 by a perfectly transparent grid (Figure 1.3). Here, x is the distance measured radially. Now, our object is to compute $\phi(x)$. For simplicity, we assume that the ratio M/m_e (ratio of the mass of positively charged ion and that of electron) is very high (infinite in the mathematical language), so that the positively charged ions do not move but form a uniform background of positive charge through which the negatively charged electrons are moving in the gas. (Since the situation is spherically symmetric, we can account for the radial variation.) Thus, we consider one-dimensional Poisson equation

$$\epsilon_0 \nabla^2 \phi = -q \quad \epsilon_0 \frac{d^2 \phi}{dx^2} = -e(n_i - n_e) \quad (1.7)$$

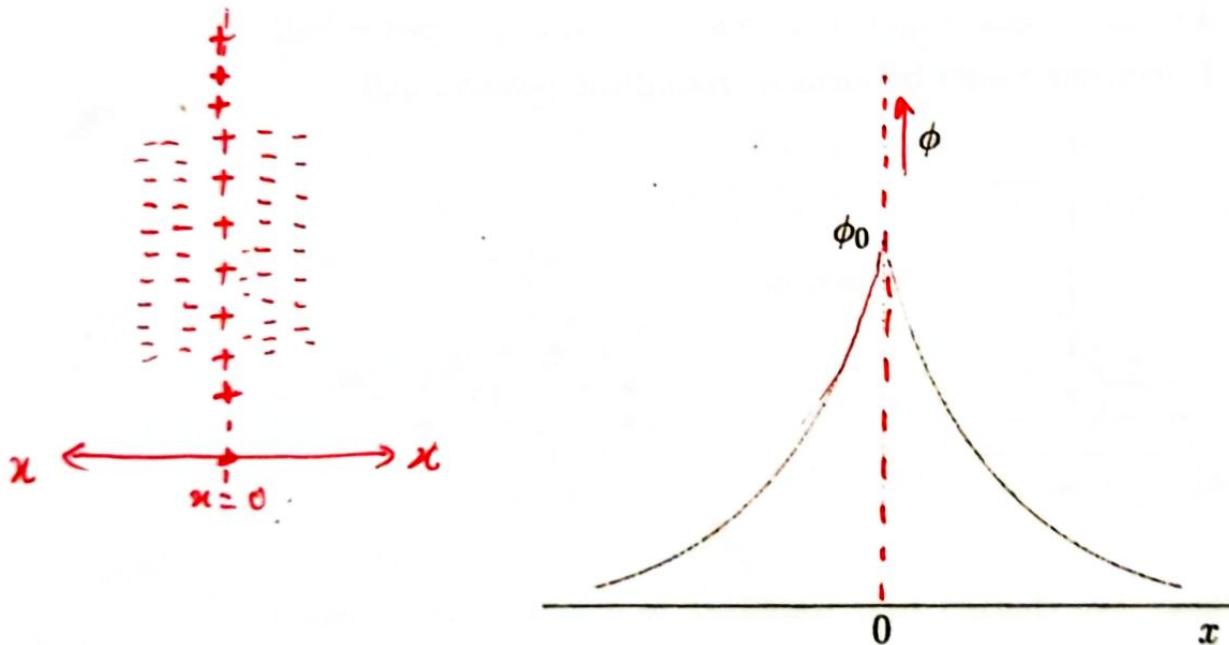


Figure 1.3: Variation of potential as a function of distance.

Here, we have accounted for the hydrogen plasma, and hence the positively charged ions are protons. At large distance, density of protons is equal to that of the electrons and is denoted by n . As the protons are not moving, the density of protons n_i is n everywhere. Thus,

$$n_i = n \quad (1.8)$$

In the presence of a potential energy $q\phi (\equiv -e\phi)$, the electron distribution function is

$$f(u) = A \exp\left[-\left(\frac{1}{2}m_e u^2 - e\phi\right)/KT_e\right] \quad (1.9)$$

where T_e is the electron temperature, u the velocity of electron. Electron density is

$$\begin{aligned} n_e &= \int_{-\infty}^{\infty} f(u) du \\ &= \int_{-\infty}^{\infty} A \exp\left[-\left(\frac{1}{2}m_e u^2 - e\phi\right)/KT_e\right] du \\ &= A \exp(e\phi/KT_e) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}m_e u^2/KT_e\right) du \\ &= A \exp(e\phi/KT_e) \sqrt{\frac{2\pi K T_e}{m_e}} \end{aligned} \quad (1.10)$$

At large distance

$$x \rightarrow \infty \quad \phi \rightarrow 0 \quad n_e \rightarrow n$$

$$(1.10) \rightarrow n = A \sqrt{\frac{2\pi K T_e}{m_e}} \quad (x \rightarrow \infty)$$

Thus, equation (1.10) becomes

$$n_e = n \exp(e\phi/KT_e) \quad (1.11)$$

Using equations (1.8), (1.9), and (1.11) in (1.7), we get

$$\epsilon_0 \frac{d^2\phi}{dx^2} = -e[n - n \exp(e\phi/KT_e)]$$

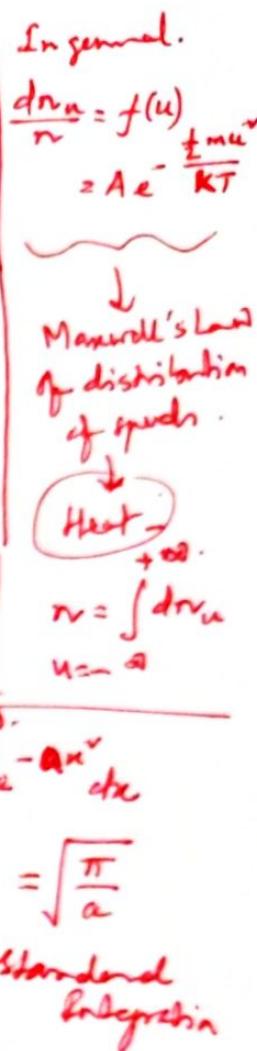
$$= en[\exp(e\phi/KT_e) - 1]$$

$$= en\left[1 + \frac{e\phi}{KT_e} + \frac{1}{2!}\left(\frac{e\phi}{KT_e}\right)^2 + \dots\right] - 1$$

$$= en\left[\frac{e\phi}{KT_e} + \frac{1}{2!}\left(\frac{e\phi}{KT_e}\right)^2 + \dots\right]$$

$$= \left(\frac{e^2 n}{KT_e}\right)\phi \left[1 + \frac{1}{2!}\left(\frac{e\phi}{KT_e}\right) + \frac{1}{3!}\cdot\left(\frac{e\phi}{KT_e}\right)^2 + \dots\right]$$

Neglected as $e, e^2, e^3 \rightarrow$ very small



Thus,

$$\epsilon_0 \frac{d^2\phi}{dx^2} = \frac{ne^2}{KT_e} \phi \quad \frac{d^2\phi}{dx^2} = \frac{\phi}{\lambda_D^2} \quad (1.12)$$

where

$$\lambda_D = \left(\frac{\epsilon_0 K T_e}{n e^2} \right)^{1/2} = \text{Const.} \times \sqrt{\frac{T_e}{n}}$$

$$D \equiv \frac{d\phi}{dx} \quad D^2 \phi = \frac{\phi}{\lambda_D^2}$$

$$\Rightarrow D = \pm \frac{1}{\lambda_D}$$

$$\therefore D \frac{d\phi}{dx} = \pm \frac{1}{\lambda_D} \phi$$

$$\Rightarrow \int \frac{d\phi}{\phi} = \int \pm \frac{1}{\lambda_D} dx$$

$$\Rightarrow \phi = e^{\pm \frac{x}{\lambda_D}}$$

General Solution of equation (1.12) is $\sqrt{\frac{K T_e}{m_e}} n_p^{-1} = \lambda_D$

$$\phi = C \exp(-|x|/\lambda_D) + D \exp(|x|/\lambda_D)$$

Second part on the right side of this equation is not feasible as it shows unphysical situation of increase of potential with the increase of distance.

Hence, the constant D is substituted equal to zero. Thus, we have

$$\phi = C \exp(-|x|/\lambda_D) \quad (1.13)$$

At $x = 0$, we have $\phi = \phi_0$. Thus, $C = \phi_0$, and equation (1.13) becomes

$$\phi = \phi_0 \exp(-|x|/\lambda_D)$$

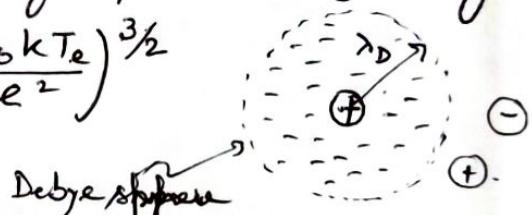
The parameter λ_D is known as the Debye length and is a measure of the shielding distance or of the thickness of the sheath. Thus, Debye length is the distance at which the potential drops by a factor e .

⊗ A necessary and obvious requirement for the existence of a plasma is that the physical dimension of a plasma system be large compared to Debye length λ_D , i.e. if L be characteristic dimension of the system, then $L \gg \lambda_D$. Otherwise, the system does not exhibit the Debye shielding effect due to lack of space.

Debye Sphere → An imaginary sphere of radius λ_D inside the plasma. At the center of the sphere, each charge in the plasma interacts collectively only with the charges that lie inside its Debye sphere, its effect on the other charges outside the sphere being effectively negligible.

The number of e^- 's inside a Debye sphere is given by

$$N_e = \left(\frac{4}{3} \pi \lambda_D^3 \right) n_e = \frac{4}{3} \frac{\pi}{\sqrt{n_e}} \left(\frac{k_B k T_e}{e^2} \right)^{3/2}$$



Shielding effect is the collective behavior of the particles inside the Debye sphere, so it is necessary that the number of e^- 's (N_e) inside the Debye sphere must be very large, i.e. $n_e \lambda_D^3 \gg 1$.

Plasma Parameter:
$$g = \frac{1}{n_e \lambda_D^3}$$

for a plasma $g \ll 1$ and this condition is called plasma approximation.

Problem :- For a gas discharge, $T = 10^4 \text{ K}$ and $n_e = 10^{16} \text{ m}^{-3}$. Calculate the Debye length.

$$\text{Soln} - \lambda_D = \left(\frac{8.85 \times 10^{-12}}{n_e e^2} \right)^{1/2} = \left[\frac{(8.85 \times 10^{-12}) \times (10^4) \times (1.38 \times 10^{-23})}{10^{16} \times (1.6 \times 10^{-19})^2} \right]^{1/2}$$

$$= 6.9 \times 10^{-5} \text{ m.}$$

Problem :- For the Earth's ionosphere, $T = 10^3 \text{ K}$ & $n_e = 10^{12} \text{ m}^{-3}$, calculate λ_D .

Problem - A welding arc with e^- concentration of $1.6 \times 10^{17} \text{ cm}^{-3}$ and $T_e = 1.3 \text{ eV}$. Find ω_p and λ_D .

$$\text{Soln} - T_e = 1.3 \text{ eV} \quad | \quad 1 \text{ eV} \approx 11,605 \text{ K} \quad | \quad 1 \text{ cm}^3 = 10^{-6} \text{ m}^3$$

$$= 1.3 \times 11,605 \text{ K}$$

$$\lambda_D = \left[\frac{(8.85 \times 10^{-12}) \times (1.38 \times 10^{-23}) \times (1.3 \times 11,605)}{(1.6 \times 10^{17} \times 10^{-6}) \times (1.6 \times 10^{-19})^2} \right]^{1/2} = 2.1 \text{ nm.}$$

$$\omega_p = \sqrt{\frac{kT_e}{M_e}} \lambda_D^{-1} = \left[\frac{1.38 \times 10^{-23} \times (1.3 \times 11,605)}{9.1 \times 10^{-31}} \right]^{1/2} \times \frac{1}{2.1 \times 10^{-9}} = 3.6 \text{ THz.}$$

PG - SBM - II

Topic - Propagation of plane EM waves in
low pressure ionised gases.

(Paper - BMT & ELECTRODYNAMICS)
(Unit - VI)

Class Note - Prepared by Arun Bhushan
(pgs- 28)

Reference :- Introduction to Plasma Physics
- Francis F. Chen.

• Propagation of plane σ -wave in low pressure

ionised gases (i.e. in rarefied plasma) with $\vec{E}_0 = 0$
 & $\vec{B}_0 = 0$ (i.e. in a field free space). ✓

A plasma is a mixture consisting of free e^- s,
 (ionised gas)
 positive ions and neutral atoms/molecules. Let n_e be
 the number of free e^- s per unit volume in the
 plasma. As $\frac{\text{positive}}{\text{ion}}$ ions are massive compared to
 the e^- s (just think about the mass ratio of e^-
 and proton $\frac{m_e}{m_p} \sim \frac{1}{2000}$), the positive ions are
 supposed to be fixed in their respective positions
 (forming a uniform background of +ve charge).

~~Assume that the background field is zero.~~
 And e^- s are distributed in such a manner that
 neutrality is maintained everywhere.

EM Waves through plasma with $B_0 = 0$

①b

- ④ $B_0 = 0 \rightarrow$ means no external force is introduced into the plasma system.

⑤ Maxwell's eqn in vacuo -

$$\vec{D} \times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t}, \quad (1a)$$

$$\vec{D} \times \vec{B}_1 = \mu_0 \epsilon_0 \frac{\partial \vec{E}_1}{\partial t} \quad (1b)$$

($\vec{J} = 0 \rightarrow$ in vacuum)

Breakdown of linearization

$a = a_0 + a_1$		
Initial value at equilibrium indicated by subscript 0		perturbation indicated by the subscript 1

Assuming plane waves varying as $e^{j(\vec{k} \cdot \vec{r} - \omega t)}$

$$\vec{D} \rightarrow j \vec{k} \cdot \vec{E}_1, \quad \frac{\partial}{\partial t} \rightarrow -j\omega.$$

$$(1a) \rightarrow j \vec{k} \times \vec{E}_1 = -(-j\omega) \vec{B}_1 \Rightarrow \vec{k} \times \vec{E}_1 = \omega \vec{B}_1 \rightarrow (2a)$$

$$(1b) \rightarrow j \vec{k} \times \vec{B}_1 = \mu_0 \epsilon_0 (-j\omega \vec{E}_1) \Rightarrow \vec{k} \times \vec{B}_1 = -j \frac{\omega}{c^2} \vec{E}_1 \rightarrow (2b).$$

$$\begin{aligned} \vec{k} \times (2a) &\Rightarrow \vec{k} \times (\vec{k} \times \vec{E}_1) = \omega (\vec{k} \times \vec{B}_1) \\ &\Rightarrow \underbrace{(\vec{k} \cdot \vec{E}_1)}_{=0} \vec{k} - \underbrace{(\vec{k} \cdot \vec{k}) \vec{E}_1}_{=k^2} = \omega \left(-j \frac{\omega}{c^2} \vec{E}_1 \right). \\ &\Rightarrow \boxed{c = \frac{\omega}{k}} - (3) \quad \left(\because \text{transverse EM wave } \vec{k} \perp \vec{E}_1 \right) \end{aligned}$$

which is the phase velocity of light.

* In a plasma with $B_0 = 0$, Maxwell's eqns - (2)

$$\vec{D} \times \vec{E}_1 = - \frac{\partial \vec{B}_1}{\partial t} \quad \rightarrow \textcircled{4a}$$

$$\vec{D} \times \vec{B}_1 = \mu_0 \vec{J}_1 + \mu_0 \epsilon_0 \frac{\partial \vec{E}_1}{\partial t}, \quad \rightarrow \textcircled{4b}.$$

$\left(\vec{J} = \vec{J}_0 + \vec{J}_1 = \vec{J}_1, (\vec{J}_0 = 0) \rightarrow \text{current density due to first order charged particle motion in presence of the EM wave.} \right)$

$$\textcircled{4a} \rightarrow j \vec{k} \times \vec{E}_1 = - (j \omega) \vec{B}_1 \Rightarrow \vec{k} \times \vec{E}_1 = \omega \vec{B}_1$$

$$\textcircled{4b} \rightarrow j \vec{k} \times \vec{B}_1 = \mu_0 \vec{J}_1 + \mu_0 \epsilon_0 (-j \omega \vec{E}_1).$$

$$\Rightarrow j \vec{k} \times \left(\frac{\vec{k} \times \vec{E}_1}{\omega} \right) = \mu_0 \vec{J}_1 = j \omega \mu_0 \epsilon_0 \vec{E}_1$$

$$\Rightarrow j \left(\frac{-k^2 \vec{E}_1}{\omega} \right) = \mu_0 \vec{J}_1 - j \mu_0 \epsilon_0 \omega \vec{E}_1 \quad (\vec{k} \cdot \vec{E}_1 = 0)$$

$$\Rightarrow \mu_0 \vec{J}_1 = j \omega \vec{E}_1 \left(-\frac{k^2}{\omega^2} + \mu_0 \epsilon_0 \right).$$

$$\Rightarrow \vec{J}_1 = j \frac{\omega \epsilon_0 \vec{E}_1}{\mu_0 \epsilon_0} \left(-\frac{k^2}{\omega^2} + \frac{1}{c^2} \right).$$

$$= j \frac{\epsilon_0 \vec{E}_1}{\omega} \left(\omega^2 - c^2 k^2 \right).$$

$$\Rightarrow \vec{J}_1 = j \frac{\epsilon_0}{\omega} (\omega^2 - c^2 k^2) \vec{E}_1 \quad \rightarrow \textcircled{5}$$

The positive ions are considered fixed. The \vec{J}_1 comes entirely from e^- motion. So,

$$\vec{J}_1 = nq \vec{v} \rightarrow \vec{J}_1 = n_0 (-e) \vec{v}_{e1}$$

$$\text{or } \vec{J}_1 = -n_0 e \vec{v}_{e1} \quad \rightarrow \textcircled{6}$$

(3)a

Eqn of motion for an e^- -

$$m_e \frac{d\vec{v}_e}{dt} = -e(\vec{E}_1 + \vec{v}_e \times \vec{B}_1)$$

(Damping due to
collisions neglected)

$$\Rightarrow m_e \left[\frac{d\vec{v}_e}{dt} + (\vec{v}_{e0} \cdot \vec{v}) \vec{v}_e \right] = -e(\vec{E}_1 + \vec{v}_e \times \vec{B}_1)$$

$$\vec{v}_e = \vec{v}_{e0} + \vec{v}_{e1}$$

\therefore Linearized e^- eqn of motion -

$$m_e \frac{d\vec{v}_{e1}}{dt} = -e\vec{E}_1 \Rightarrow \vec{v}_{e1} = -j \frac{e}{m_e \omega} \vec{E}_1 \rightarrow (7)$$

$$\begin{aligned} \bullet (\vec{v}_e \cdot \vec{v})(\vec{v}_e) &= [(\vec{v}_{e0} + \vec{v}_{e1}) \cdot \vec{v}] (\vec{v}_{e0} + \vec{v}_{e1}) \\ \bullet \vec{v}_e \times \vec{B}_1 &= \vec{v}_{e1} \times \vec{B}_1 \approx 0. \end{aligned}$$

$\vec{v}_0 = 0; |\vec{v}_{e1}| \rightarrow \text{is very small}$
 $\text{so, quadratic terms are negligible.}$

Combining eqns (5), (6) & (7) \rightarrow .

$$j \frac{\epsilon_0}{m_e} (\tilde{\omega}^2 - c^2 k^2) \vec{E} = -n_0 e v_{e1} = j \frac{n_0 e^2}{m_e \omega} \vec{E}$$

$$\Rightarrow \tilde{\omega}^2 - c^2 k^2 = \frac{n_0 e^2}{m_e \epsilon_0} = \omega_p^2, \quad | \omega_p \rightarrow \text{plasma freqy.}$$

$$\Rightarrow \boxed{\tilde{\omega}^2 = \omega_p^2 + c^2 k^2} \rightarrow (8)$$

Which is the dispersion relation of e.m wave propagating through cold plasma with $B_0 = 0$ (with no dc magnetic field).

(Apply the procedure of linearization to derive the dispersion relation.)

Eqn (8) is called dispersion relation for TEM waves propagating through rarefied plasma, when $B_0 = 0$

Phase velocity of TEM through plasma -

$$v_p = \frac{\omega}{k} = \sqrt{\frac{\omega_p^2}{k^2} + c^2} \Rightarrow \omega_p = c \sqrt{1 + \frac{\omega_p^2 c^2}{k^2}} \quad (9)$$

$\therefore v_p > c$, i.e. phase velocity of the TEM wave through plasma is greater than speed of light c .

Let us check the group velocity -

$$v_g = \frac{dw}{dk} = \frac{c^2 k}{\omega} \quad \left| \begin{array}{l} \because \omega^2 = \omega_p^2 + c^2 k^2 \\ \Rightarrow 2\omega d\omega = 0 + c^2 2k dk \end{array} \right.$$

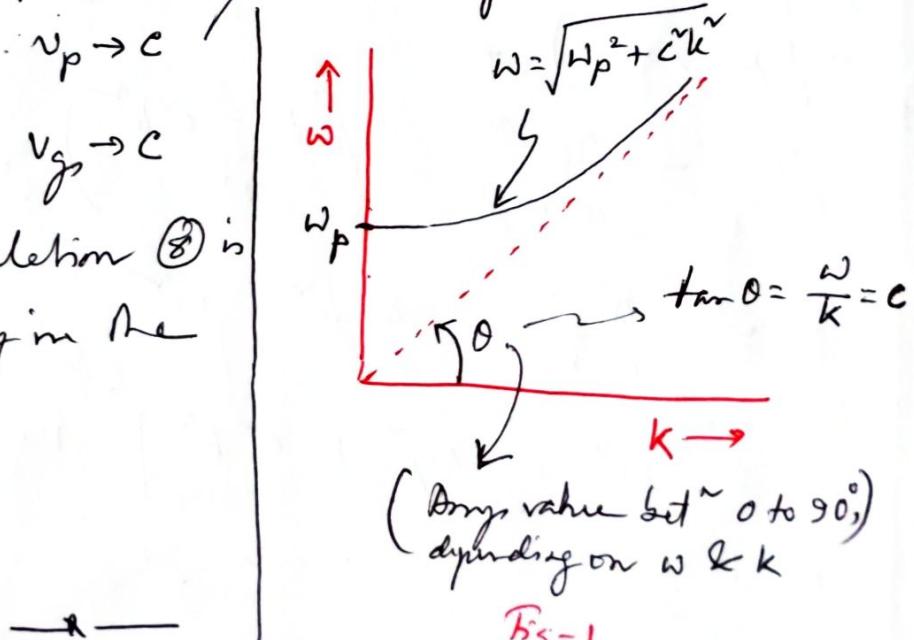
$$\boxed{v_g = c \left(1 + \frac{\omega_p^2 c^2}{k^2} \right)^{-\frac{1}{2}}} < c. \quad (10)$$

Thus, the group velocity is always smaller than c .

$$\text{With } k \rightarrow \infty \Rightarrow v_p \rightarrow c$$

$$\therefore v_g \rightarrow c$$

The dispersion relation (8) is shown graphically in the fig - 1.



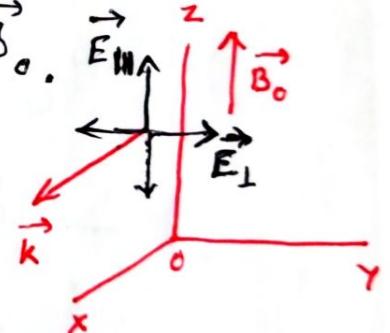
EM Waves in a cold rarefied plasma with a uniform magnetic field \vec{B}_0 . (i.e. cold rarefied magnetoplasma) (4)

For the sake of simplification, we ignore the e^-e^- and e^- -ion collisions and the thermal motion of the e^- s ($kT_e \approx 0$). The ions are considered as fixed, (being much heavier) and so, only the e^- s are influenced by the E and B field of the incident EM wave. The EM wave is a transverse EM wave with \vec{k} , \vec{E} and \vec{B} mutually \perp^r to each other. We consider the two cases -

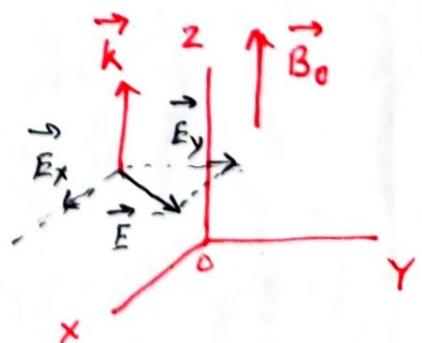
①. The EM wave is propagating \perp^r to the ambient magnetic field, i.e. $\vec{k} \perp \vec{B}_0$.

② electric field \vec{E}_{\parallel} parallel to \vec{B}_0

③ electric field \vec{E}_{\perp} \perp^r to \vec{B}_0



④. The EM wave is propagating \parallel to the ambient magnetic field.



Case 1a :- $\vec{k} \perp \vec{B}_0$ and $(\vec{E}) \parallel \vec{B}_0$ (ordinary wave)

By Maxwell's eqns in the plasma -

$$\vec{\nabla} \times \vec{E}_1 = - \frac{\partial \vec{B}_1}{\partial t} \quad \text{--- (1a)} \quad \vec{\nabla} \times \vec{B}_1 = \mu_0 \vec{J}_1 + \mu_0 \epsilon_0 \frac{\partial \vec{E}_1}{\partial t} \quad \text{--- (1b)}$$

EOM wave is assumed to be plane wave varying as $e^{j(\vec{k} \cdot \vec{r} - \omega t)}$. So, $\vec{\nabla} \rightarrow j \vec{k}$ and $\frac{\partial}{\partial t} \rightarrow -j\omega$.

$$(1a) \rightarrow j \vec{k} \times \vec{E}_1 = j \omega \vec{B}_1 \Rightarrow \vec{k} \times \vec{E}_1 = \omega \vec{B}_1 \quad \text{--- (2a)}$$

$$(1b) \rightarrow j \vec{k} \times \vec{B}_1 = \mu_0 \vec{J}_1 + (-j) \mu_0 \epsilon_0 \omega \vec{E}_1 \quad \text{--- (2b)}$$

$$\Rightarrow j \vec{k} \times \left(\frac{\vec{k} \times \vec{E}_1}{\omega} \right) = \mu_0 \vec{J}_1 - j \mu_0 \epsilon_0 \omega \vec{E}_1 \quad (\text{applying (2a)})$$

$$\Rightarrow -j \frac{\vec{k}^2 \vec{E}_1}{\omega} = \mu_0 \vec{J}_1 - j \frac{\omega}{c^2} \vec{E}_1 \quad \begin{array}{l} \vec{k} \perp \vec{E} \rightarrow \vec{k} \cdot \vec{E} = 0 \\ \therefore \vec{k} \times (\vec{k} \times \vec{E}) = (\vec{k} \cdot \vec{E}) \vec{k} - (\vec{k} \cdot \vec{k}) \vec{E} \end{array}$$

$$\Rightarrow -j k^2 c^2 \vec{E}_1 = \mu_0 \omega c^2 \vec{J}_1 - j \omega \vec{E}_1$$

$$\Rightarrow (\omega^2 - k^2 c^2) \vec{E}_1 = -j \frac{\omega}{\epsilon_0} \vec{J}_1 \quad \text{--- (3)}$$

(Here $k = k_x$, $\vec{E}_1 \rightarrow \vec{E}_{z1}$, $\vec{J}_1 \rightarrow -\vec{J}_{z1}$)

$$|\vec{B}_0| \gg |\vec{B}_1|$$

$$m_e \frac{d\vec{v}_e}{dt} = -e \left(\vec{E}_1 + \vec{v}_e \times \vec{B}_0 \right) \quad \Rightarrow (\hat{k} \vec{v}_{z1} \times \hat{k} \vec{B}_0) = 0.$$

$$\Rightarrow m_e \left[\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \vec{\nabla}) \vec{v}_e \right] = -e \left(\vec{E}_1 + \vec{v}_e \times \vec{B}_0 \right)$$

$$\Rightarrow m_e \frac{\partial \vec{v}_{e1}}{\partial t} = -e \vec{E}_1 \quad \begin{array}{l} (\text{using the problem of}) \\ \text{linearisation.} \end{array}$$

$$\vec{v}_e = \vec{v}_{e0} + \vec{v}_{e1}$$

$$a_0 \rightarrow a = a_0 + a_1$$

$a_1 \rightarrow \text{perturbation}$

and $\vec{v}_{e0} \approx 0$ (no Diamagnetic motion).

(6)a

$$(4) \rightarrow m_e(-j\omega \vec{v}_{e1}) = -e\vec{E}_1$$

$$\Rightarrow \vec{v}_{e1} = -j \frac{e}{m_e \omega} \vec{E}_1 \quad \rightarrow (5).$$

$$\therefore \vec{J} = n_g \vec{v} \rightarrow \vec{J}_o + \vec{J}_i = (n_o + n_{o1})(-e)(\vec{v}_{eo} + \vec{v}_{e1})$$

$$\Rightarrow \vec{J}_i = -n_o e \vec{v}_{e1} \rightarrow (6)$$

(3), (5), (6) combining these eqns -

$$(\omega^2 - c^2 k^2) \vec{E}_1 = -j \frac{\omega}{\epsilon_0} (-n_o e \vec{v}_{e1})$$

$$= j \frac{\omega}{\epsilon_0} n_o e \left(-j \frac{e}{m_e \omega} \vec{E}_1 \right).$$

$$\Rightarrow \omega^2 - c^2 k^2 = \frac{n_o e^2}{\epsilon_0 m_e} = \omega_p^2, \quad \omega_p \rightarrow \text{plasma frequency}$$

$$\Rightarrow \boxed{\omega^2 = \omega_p^2 + c^2 k^2} \quad \rightarrow (7).$$

Which is the dispersion relation and same as that when $B_0 = 0$. It means that the magnetic field \vec{B}_0 could not produce any influence on the propagation of EM wave, as compared to the case without an ambient magnetic field. Therefore, this EM wave is called 'ordinary' EM wave.

(6) b

Dispersion relation for O-wave -

$$\omega^2 = \omega_p^2 + c^2 k^2 \quad \text{by eqn (7).}$$

$$\Rightarrow \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\Rightarrow \frac{\omega_p^2}{c^2} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1} \left(= \frac{1}{n}\right), \quad n \rightarrow \text{r.i.}$$

Cut off frequency: \rightarrow It is frequency when r.i. goes to zero. Thus, it divides the dispersion diag. into propagation and nonpropagation region.
 when $n \rightarrow 0$
 $k \rightarrow 0$. } $\rightarrow \omega = \omega_p$.

Nonpropagation region ($\frac{\omega_p^2}{c^2} < 0$) for the frequencies range for $\left(1 - \frac{\omega_p^2}{\omega^2}\right) < 0 \rightarrow \omega < \omega_p$.

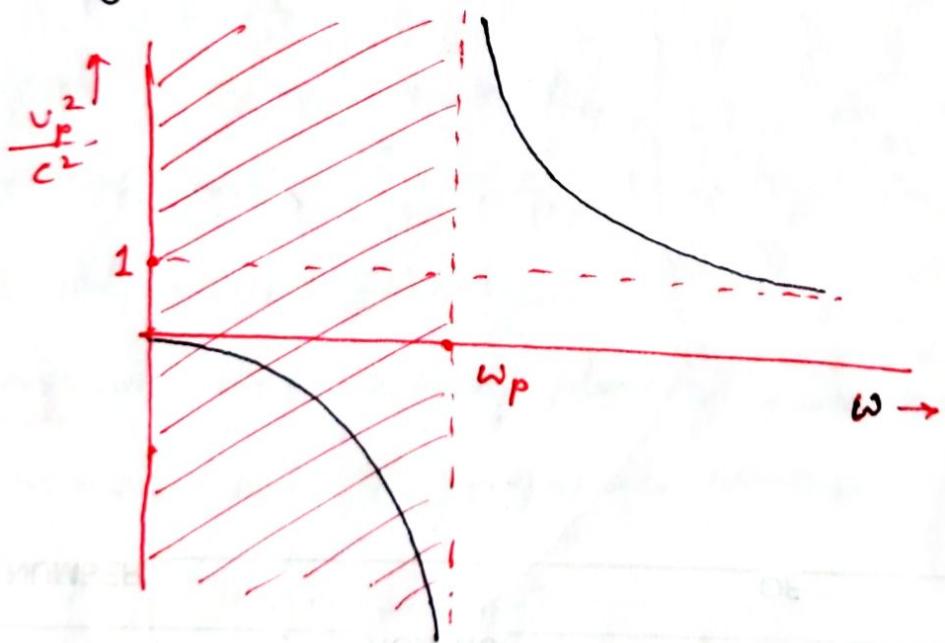


Fig- $\frac{\omega_p^2}{c^2}$ vs ω diagram for O-wave. Shaded region for frequency range 0 to ω_p is the nonpropagation region for O-wave.

Ques - (16) - $\vec{k} \perp \vec{B}_0$ and $(\vec{E}_{\text{em}}) \perp \vec{B}_0$ (Extraordinary wave) (7)

To treat this case, we suppose that

$$\vec{B}_0 = \hat{i} B_0 \rightarrow \text{constant magnetic field.}$$

$\vec{k} = \hat{i} k_x$. → Along the direction of propagation of EM-wave

$$k \vec{E}_1 = \hat{j} E_1 \rightarrow \text{Dir^n of } \vec{E} \text{ field of EM wave.}$$

But as the EM wave propagates through the plasma, it develops a component (E_x) along \vec{k} , i.e. along the direction of propagation of the EM wave (along +x-direction).

So, \vec{E}_1 must have both x and y components:

$$\vec{E}_1 = \hat{i} E_{1x} + \hat{j} E_{1y}.$$

It turns out that the EM waves with $\vec{E}_1 \perp \vec{B}_0$

tend to be elliptically polarised instead of plane

polarised as they propagate through the plasma.

B_2 - The E-field is elliptically polarised for an extraordinary wave.

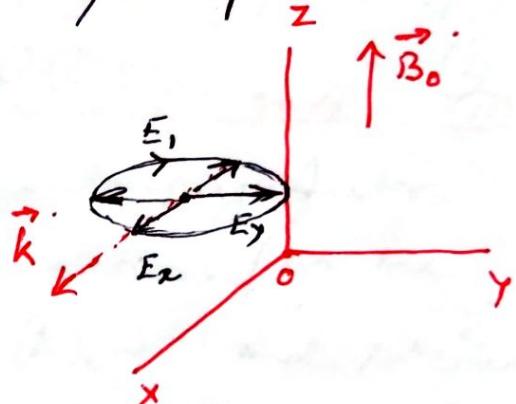
By Maxwell's eqn -

$$\vec{\nabla} \times \vec{E}_1 = - \frac{\partial \vec{B}_1}{\partial t} \quad \text{--- (1A)}$$

$$\vec{\nabla} \times \vec{B}_1 = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial \vec{E}_1}{\partial t} \rightarrow (1B)$$

Assuming \vec{E} and \vec{B} varying as $e^{j(\vec{k} \cdot \vec{r} - \omega t)}$,

$$\vec{\nabla} \rightarrow j\vec{k}, \quad \frac{\partial}{\partial t} \rightarrow -j\omega.$$



(8)

$$(1a) \rightarrow j \vec{k} \times \vec{E}_1 = - (j\omega) \vec{B}_1, \Rightarrow \vec{k} \times \vec{E}_1 = \omega \vec{B}_1.$$

$$(1b) \rightarrow j \vec{k} \times \vec{B}_1 = \mu_0 \vec{J}_1 + (-j\omega) \mu_0 \epsilon_0 \vec{E}_1$$

$$\Rightarrow j \vec{k} \times \left(\frac{\vec{k} \times \vec{E}_1}{\omega} \right) = \mu_0 \vec{J}_1 - j \omega \mu_0 \epsilon_0 \vec{E}_1$$

$$\Rightarrow j \left[(\vec{k} \cdot \vec{E}_1) \vec{k} - \underbrace{(\vec{k} \cdot \vec{k}) \vec{E}_1}_{= k^2} \right] = \mu_0 \omega \vec{J}_1 - j \omega^2 \mu_0 \epsilon_0 \vec{E}_1$$

$$ik_x \cdot (i \hat{E}_{1x} + j \hat{E}_{1y}) = k_x E_{1x}$$

$$\Rightarrow j (k_x E_{1x} \vec{k} - k_x^2 \vec{E}_1) = \mu_0 \omega \vec{J}_1 - j \omega^2 \mu_0 \epsilon_0 \vec{E}_1$$

$$\Rightarrow (\omega^2 - c^2 k_x^2) \vec{E}_1 + c^2 k_x E_{1x} \vec{k} = -j \mu_0 \omega c^2 \vec{J}_1$$

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Eqs of motion of an e^- ignoring damping factor due to collisions and thermal motion (as the plasma is assumed to be cold ($kT_e \sim 0$) and rarefied).

$$m_e \frac{d\vec{v}_e}{dt} = -e \left(\vec{E}_1 + \vec{v}_e \times \vec{B}_0 \right) \quad (|\vec{B}_0| + |\vec{B}_1| \approx |\vec{B}_0|).$$

$$\Rightarrow m_e \left[\frac{d\vec{v}_e}{dt} + (\vec{v}_e \cdot \vec{v}) \vec{v}_e \right] = -e \left(\vec{E}_1 + \vec{v}_e \times \vec{B}_0 \right).$$

\vec{v}_{e1} is very small, so $(\vec{v}_{e1} \cdot \vec{v}) \vec{v}_{e1}$ neglected and.

$$\vec{v}_{e0} = 0 \quad \text{in } \vec{v}_e = \vec{v}_{e0} + \vec{v}_{e1}$$

$$m_e \frac{d\vec{v}_{e1}}{dt} = -e \left(\vec{E}_1 + \vec{v}_{e1} \times \vec{B}_0 \right) \quad (|\vec{B}_0| \gg |\vec{B}_1|).$$

(9)

$$m_e \frac{d}{dt} (\hat{i} v_{e1x} + \hat{j} v_{e1y}) = -e \left[\hat{i} \underline{E}_{1x} + \hat{j} \underline{E}_{1y} \right] \\ + \left[\hat{i} \underline{v}_{e1x} + \hat{j} \underline{v}_{e1y} \right] \times \hat{k} B_0$$

(e^- 's are drifted by the \vec{E} , field on x-y plane along $-\vec{E}_1$, so \vec{v}_{e1} will have two components)

$$\text{or } m_e \frac{d}{dt} (\hat{i} v_{e1x} + \hat{j} v_{e1y}) = -e \left[\hat{i} \underline{E}_{1x} + \hat{j} \underline{E}_{1y} \right] \\ + (-j) \underline{v}_{e1x} B_0 + \hat{i} \underline{v}_{e1y} B_0$$

Decomposing the above eqn component wise -

$$m_e (-j\omega) \underline{v}_{e1x} = -e \left[\underline{E}_{1x} + \underline{v}_{e1y} B_0 \right] \quad \left(\frac{d}{dt} \rightarrow -j\omega \right)$$

$$\& m_e (-j\omega) \underline{v}_{e1y} = -e \left[\underline{E}_{1y} - \underline{v}_{e1x} B_0 \right]$$

$$\therefore \underline{v}_{e1x} = -j \frac{e}{m_e \omega} \left[\underline{E}_{1x} + \underline{v}_{e1y} B_0 \right] \quad \rightarrow (6a)$$

$$\text{and } \underline{v}_{e1y} = -j \frac{e}{m_e \omega} \left[\underline{E}_{1y} - \underline{v}_{e1x} B_0 \right] \quad \rightarrow (6b)$$

Putting 6b in 6a \rightarrow

$$v_{e1x} = \left(-j \frac{e}{m_e \omega} \right) \left[\underline{E}_{1x} + \left(-j \frac{e}{m_e \omega} \right) \left\{ \underline{E}_{1y} - \underline{v}_{e1x} B_0 \right\} B_0 \right]$$

$$\Rightarrow v_{e1x} \left[1 + \left(-j \frac{e}{m_e \omega} \right)^2 B_0^2 \right] = \left(-j \frac{e}{m_e \omega} \right) \underline{E}_{1x} + \left(-j \frac{e}{m_e \omega} \right)^2 \underline{E}_{1y} B_0$$

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(1D)

$$\therefore \vec{J} = q n \vec{v} \rightarrow \vec{J}_i = -en_0 \vec{v}_{ei} \rightarrow (3)$$

$$\vec{J} \rightarrow \vec{J}_0 + \vec{J}_i = \vec{J}_i \quad (\because \vec{J}_0 = 0).$$

$$n_0 \rightarrow n_0 + n_{0i} \approx n_0.$$

$$\vec{v}_0 \rightarrow \vec{v}_0 + \vec{v}_{0i} = \vec{v}_{0i} \quad (\because \vec{v}_0 = 0).$$

(3) in

$$(2) \Rightarrow (\omega^2 - c^2 k_x^2) \vec{E}_i + c^2 k_x E_{ix} \vec{k} = j \mu_0 \epsilon_0 c^2 e n_0 \vec{v}_{ei} \rightarrow (4)$$

$$\vec{E}_i = i \hat{E}_{ix} + j \hat{E}_{iy} \quad \vec{k} = i \hat{k}_x \quad \vec{v}_{ei} = \left(\hat{i} v_{eix} + \hat{j} v_{eiy} \right)$$

Separating (4) into x and y components -

$$(\omega^2 - c^2 k_x^2) E_{ix} + c^2 k_x E_{ix} k_x = j \frac{\mu_0 e n_0}{\epsilon_0} v_{eix}$$

$$(\omega^2 - c^2 k_x^2) E_{iy} = j \frac{\mu_0 e n_0}{\epsilon_0} v_{eiy}$$

$$\therefore (\omega^2 - c^2 k_x^2) E_{ix} = j \frac{\mu_0 e n_0}{\epsilon_0} v_{eix} \rightarrow (5a)$$

$$\text{and } (\omega^2 - c^2 k_x^2) E_{iy} = j \frac{\mu_0 e n_0}{\epsilon_0} v_{eiy} \rightarrow (5b)$$

go to page - 8.

from page 9

$$\Rightarrow v_{eix} \left[1 - \omega_c^2/\omega^2 \right] = \frac{e}{m_e \omega} \left[-j E_{ix} - \frac{\omega_c}{\omega} E_{iy} \right]$$

$$\Rightarrow v_{eix} = \frac{e}{m_e \omega} \left[-j E_{ix} - \frac{\omega_c}{\omega} E_{iy} \right] \left(1 - \frac{\omega_c^2}{\omega^2} \right)^{-1} \rightarrow (7a)$$

$$\text{where } \omega_c = \frac{e B_0}{m_e} \rightarrow \text{cyclotron frequency} \quad \left(\because \frac{m v^2}{r} = q v B \right)$$

$$\Rightarrow m \omega_c = q B.$$

Now putting (6a) in (6b) →

(1)

$$\boxed{\text{H.H.}} \quad v_{a,y} = \frac{e}{m_e \omega} \left[-j E_{1,y} + \frac{\omega_c}{\omega} E_{1,x} \right] \left(1 - \frac{\omega_c^2}{\omega^2} \right)^{-1} \rightarrow (7b)$$

likewise here expressions of v_{R1x} and $v_{a,y}$ in eqn

(5a) & (5b) respectively -

$$(7a) \text{ in } (5a) \rightarrow \underline{\omega^2 E_{1,x}} = j \cancel{\frac{\omega/e m_0}{\epsilon_0}} \cancel{\frac{e}{m_e \omega}} \left[-j E_{1,x} - \frac{\omega_c}{\omega} E_{1,y} \right] \left(1 - \frac{\omega_c^2}{\omega^2} \right)^{-1} \quad \left. \begin{array}{l} \omega_p^2 \sqrt{n_0 e^2 / m_e \epsilon_0} \\ \text{Plasma freq} \end{array} \right\}$$

$$= \left[\omega_p^2 E_{1,x} - j \frac{\omega_c \omega_p^2}{\omega} E_{1,y} \right] \left(1 - \frac{\omega_c^2}{\omega^2} \right)^{-1}$$

$$\rightarrow \underbrace{\left[\omega^2 \left(1 - \frac{\omega_c^2}{\omega^2} \right) - \omega_p^2 \right]}_{(A)} E_{1,x} + \underbrace{\left(j \frac{\omega_c \omega_p^2}{\omega} \right)}_{(B)} E_{1,y} = 0. \rightarrow (8a)$$

(7b) in (5b) →

$$\boxed{\text{H.H.}} \quad \left[\left(\omega^2 - c^2 k_a^2 \right) \left(1 - \frac{\omega_c^2}{\omega^2} \right) - \omega_p^2 \right] E_{1,y} + \underbrace{\left(-j \frac{\omega_c \omega_p^2}{\omega} \right)}_{(C)} E_{1,x} = 0. \rightarrow (8b)$$

(8a) & (8b) are two simultaneous eqns for $E_{1,x}$ and $E_{1,y}$ which are compatible only when the determinant of the coefficients vanishes, i.e.

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0. \quad \rightarrow (9).$$

(12)

(8a) & (8b) combining \rightarrow

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E_{1x} \\ E_{1y} \end{pmatrix} = 0 \quad \rightarrow \quad (10).$$

Introducing upper hybrid frequency $\omega_h = \sqrt{\omega_p + \omega_c}$ -

$$A = (\omega^2 - \omega_h^2)$$

$$A = \omega^2 - c^2 k_x^2 - \frac{\omega_c^2}{\omega^2} + c^2 k_x^2 \frac{\omega_c^2}{\omega^2} - \omega_p^2$$

$$\therefore D = \left[\omega^2 - \omega_h^2 - c^2 k_x^2 \left(1 - \frac{\omega_c^2}{\omega^2} \right) \right]$$

Condition in (9) given $AD = BC$, i.e.

$$(\omega^2 - \omega_h^2) \left[\omega^2 - \omega_h^2 - c^2 k_x^2 \left(1 - \frac{\omega_c^2}{\omega^2} \right) \right] = \left(\frac{\omega_p \omega_c}{\omega} \right)^2$$

$$\Rightarrow c^2 k_x^2 \left(1 - \frac{\omega_c^2}{\omega^2} \right) = (\omega^2 - \omega_h^2) - \left(\frac{\omega_p \omega_c}{\omega} \right)^2 (\omega^2 - \omega_h^2)^{-1}$$

$$\Rightarrow \frac{c^2 k_x^2}{\omega^2} = \frac{(\omega^2 - \omega_h^2) - \left(\frac{\omega_p \omega_c}{\omega} \right)^2 (\omega^2 - \omega_h^2)^{-1}}{\omega^2 - \omega_c^2}$$

$$= 1 - \frac{\omega_p^2 (\omega^2 - \omega_h^2) + \left(\frac{\omega_p \omega_c}{\omega} \right)^2}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_h^2)}$$

$$\left| \begin{array}{l} \omega^2 - \omega_h^2 \\ = \omega^2 - \omega_c^2 - (\omega_h^2 - \omega_c^2) \\ = (\omega^2 - \omega_c^2) - \omega_p^2 \end{array} \right.$$

$$= 1 - \frac{\omega_p^2 (\omega^2 - \omega_h^2) + \omega_p^2 \omega_c^2}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_h^2)}$$

$$\Rightarrow \frac{c^2 k_x^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega^2 (\omega^2 - \omega_c^2 - \omega_p^2) + \omega_p^2 \omega_c^2}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_h^2)}$$

$$\Rightarrow \boxed{\frac{c^2 k_x^2}{\omega^2} = \frac{c^2}{v_p^2} = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2} = n} \quad \rightarrow (11)$$

$$\left| \begin{array}{l} v_p \rightarrow \text{phase velocity} \\ k = k_x \\ n = \text{r.i. of plasma} \end{array} \right.$$

which is the dispersion relation for the EM wave propagating through a cold ionized plasma with an ambient magnetic field

(13)

field \vec{B}_0 and electric field \vec{E} of 1^{st} to B_0 on incidence.

This wave in its progression through plane behaves as an EM wave, partly transverse and partly longitudinal and it is called Extraordinary wave.

① Cut off frequency! - Cut off frequency occurs when the r.i. goes to zero or when wavelength becomes ∞ (or $k \rightarrow 0$). By eqn (11) →

$$I = \frac{\omega_p}{\omega} \cdot \frac{\omega - \omega_p}{\omega - \omega_\lambda^2}$$

$$= \frac{\omega_p}{\omega} \cdot \frac{1}{1 - \omega_c^2 / (\omega^2 - \omega_p^2)}$$

$$\Rightarrow 1 - \frac{\omega_c^2}{\omega^2 - \omega_p^2} = \frac{\omega_p}{\omega}$$

$$\Rightarrow 1 - \frac{\omega_p}{\omega} = \frac{\omega_c^2}{\omega^2 - \omega_p^2}$$

$$\Rightarrow \frac{\omega_c^2 / \omega}{1 - \omega_p^2 / \omega^2}$$

$$\Rightarrow \left(1 - \frac{\omega_p}{\omega}\right)^2 = \frac{\omega_c^2}{\omega^2} \Rightarrow 1 - \frac{\omega_p}{\omega} = \pm \frac{\omega_c}{\omega}$$

$$\Rightarrow \omega^2 \mp \omega_c \omega - \omega_p^2 = 0. \quad \rightarrow (12)$$

Which is a quadratic eqn for ω .

$$(ax^2 + bx + c = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a})$$

(14)

$$\therefore \omega = \frac{-(\mp \omega_c) \pm \sqrt{(\mp \omega_c)^2 - 4 \times 1 \times (-\omega_p^2)}}{2 \times 1}$$

$$= \frac{1}{2} \left[\pm \omega_c \pm \sqrt{\omega_c^2 + 4 \omega_p^2} \right]$$

$$\text{or } \frac{\omega_R^2 \mp \frac{1}{2} \left[\omega_c + \sqrt{\omega_c^2 + 4 \omega_p^2} \right]}{\omega_L^2 \mp \frac{1}{2} \left[-\omega_c + \sqrt{\omega_c^2 + 4 \omega_p^2} \right]}$$

(13a)

(13b)

 $\omega_L < \omega_p$

In both cases the sign in front of ω_c square root is taken, so that ω remains +ve. ω_R and ω_L are called the right-hand and left-hand cutoffs. A wave is generally reflected at a cutoff.

Resonance of the extraordinary wave :-

It is found by setting k equal to ∞ . Resonance occurs when ω , i.e., between ω , when the wavelength becomes zero.

For any finite ω , $k \rightarrow \infty$ implies $\omega \rightarrow \omega_h$.

$$\omega = \frac{\omega^2 (\omega^2 - \omega_h^2) - \omega_p^2 (\omega^2 - \omega_p^2)}{\omega^2 (\omega^2 - \omega_h^2)}$$

$$\Rightarrow \omega^2 (\omega^2 - \omega_h^2) = 0.$$

$$\Rightarrow \omega^2 = \omega_h^2 \quad (\because \omega \neq 0)$$

$$\text{or } \omega^2 = \omega_h^2 = \omega_p^2 + \omega_L^2 \rightarrow \text{At the resonance.}$$

which is similar to the dispersion relation for electrostatic electron waves across \vec{B}_0 .

[When $kT_e > 0$, thermal motion is considerable and e's are streaming into adjacent layers of plasma with their

thin thermal velocities and thus, rays will carry information (to the adjacent layer) about what is happening in the ~~region~~ oscillating region. The plasma oscillation is then properly called plasma wave. Since ions are assumed to be at rest, this plasma wave can be characterised as electron plasma wave. (Reed - the local plasma frequency)

As the a wave of given ω approaches the resonant point ($\omega \rightarrow \omega_h$), the wave energy is converted into upper hybrid oscillations. The EM wave is partly electromagnetic and partly electrostatic. At resonance, this wave loses its electromagnetic character and becomes an electrostatic oscillation.

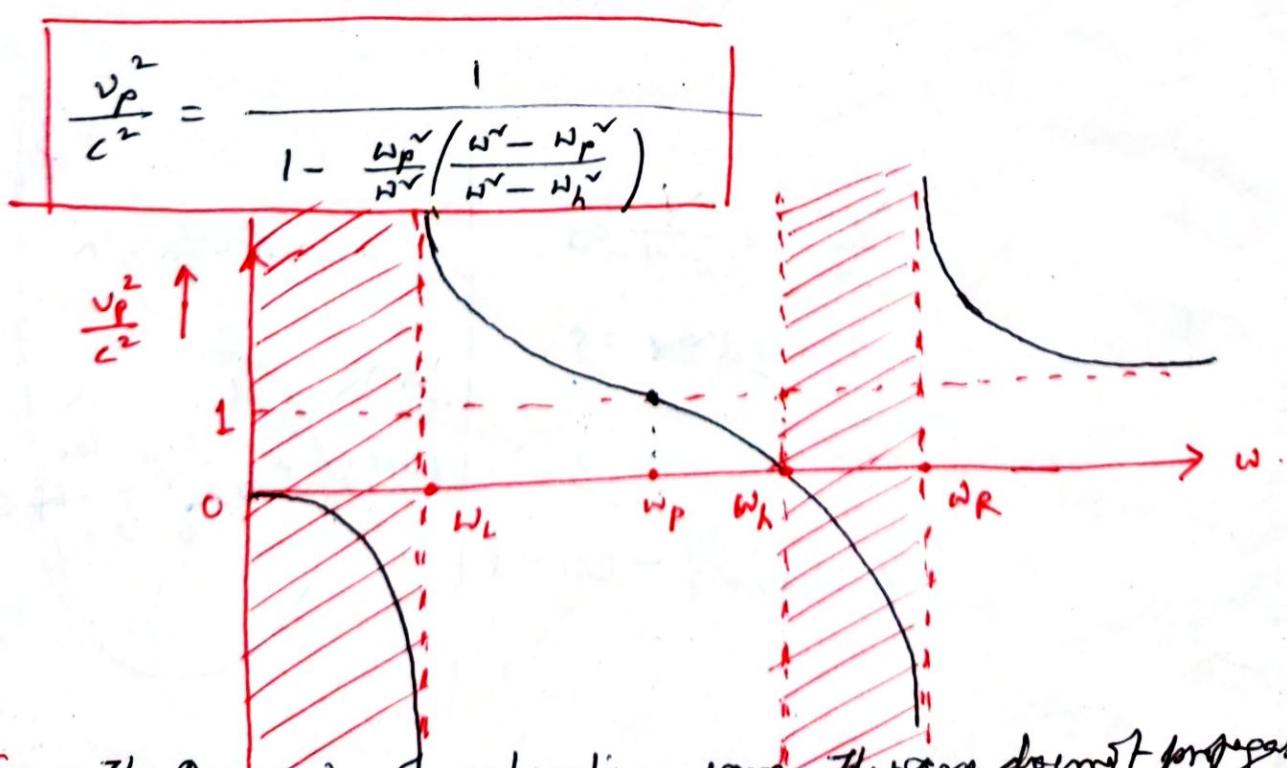


Fig - The Dispersion of extraordinary wave. The wave doesn't propagate in the shaded region.

(16)

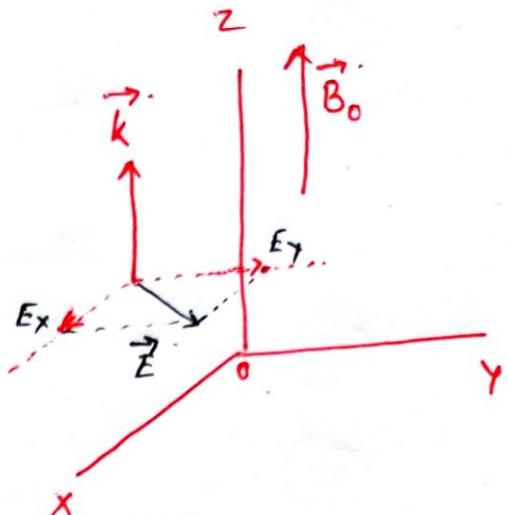
The cut-off and resonance frequencies (ω_R , ω_L & ω_h) divide the dispersion diagram into regions of propagation and nonpropagation. Instead of ω - k diagram, here we are using $\frac{v_p}{c}$ vs. ω (or $\frac{1}{n_r}$ vs ω) diagram which is more enlightening for extraordinary waves. At large ω , $v_p \rightarrow c$, and hence $\frac{v_p}{c} \rightarrow 1$. As $\omega \rightarrow \omega_R$ (right-hand cut-off), $v_p \rightarrow \infty$. Between the $\omega = \omega_R$ and $\omega = \omega_h$ layer, $\frac{\omega}{k_r} = v_p$ is -ve, and there is no propagation possible. At $\omega = \omega_h$, $\frac{v_p}{c} = 0$ and at $\omega = \omega_p$, $\frac{v_p}{c} = 1$ and as $\omega \rightarrow \omega_L$, the ratio $\frac{v_p}{c}$ and so, v_p increase beyond c and becomes to ∞ . For $\omega < \omega_L$, there is another region of nonpropagation. The extraordinary wave, therefore, has two regions of propagation (ω_L to ω_R) and ($\omega_R \rightarrow \infty$) separated by a stop band (ω_h to ω_R).

Case-2- EM waves propagating parallel to \vec{B}_0 (i.e. $\vec{k} \parallel \vec{B}_0$)

$$\vec{B}_0 = \hat{k} B_0.$$

$$\vec{k} = \hat{k} k$$

$$\vec{E} = \hat{i} E_x + \hat{j} E_y$$



By Maxwell's eqn -

$$\vec{\nabla} \times \vec{E}_1 = - \frac{\partial \vec{B}_1}{\partial t} \rightarrow (1a)$$

$$\vec{\nabla} \times \vec{B}_1 = \mu_0 \vec{J}_1 + \mu_0 \epsilon_0 \frac{\partial \vec{E}_1}{\partial t} \rightarrow (1b)$$

Assuming \vec{E} and \vec{B} varying as $e^{j(\vec{k} \cdot \vec{r} - \omega t)}$,

$$\vec{\nabla} \rightarrow j \vec{k} \quad \text{and} \quad \frac{\partial}{\partial t} \rightarrow -j\omega.$$

$$\therefore (1a) \Rightarrow j \vec{k} \times \vec{E}_1 = -(-j\omega) \vec{B}_1 \Rightarrow \vec{k} \times \vec{E}_1 = \omega \vec{B}_1,$$

$$(1b) \Rightarrow j \vec{k} \times \vec{B}_1 = \mu_0 \vec{J}_1 + \mu_0 \epsilon_0 (-j\omega) \vec{E}_1$$

$$\Rightarrow j \vec{k} \times \left(\frac{\vec{k} \times \vec{E}_1}{\omega} \right) = \mu_0 \vec{J}_1 - j \mu_0 \epsilon_0 \omega \vec{E}_1$$

$$\Rightarrow j \left[(\vec{k} \cdot \vec{E}_1) \vec{k} - (\vec{k} \cdot \vec{k}) \vec{E}_1 \right] = \omega \mu_0 \vec{J}_1 - j \mu_0 \epsilon_0 \omega \vec{E}_1$$

$$\Rightarrow j (0 - k^2 \vec{E}_1) = \omega \mu_0 \vec{J}_1 - j \mu_0 \epsilon_0 \omega \vec{E}_1$$

$$\Rightarrow j (\mu_0 \epsilon_0 \omega \vec{E}_1 - k^2 \vec{E}_1) = \omega \mu_0 \vec{J}_1$$

$$\Rightarrow (n^2 - c^2 k^2) \vec{E}_1 = -j c^2 \omega \mu_0 \vec{J}_1 \rightarrow (2).$$

$$\therefore \vec{J}_1 = n \epsilon_0 \omega \vec{E}_1 \quad \left| \begin{array}{l} \vec{v}_e = \vec{v}_{eo} + \vec{v}_{ei} \\ = -n \epsilon_0 \omega \vec{v}_{ei} \end{array} \right. \rightarrow (3).$$

Putting ③ in ② →

(18)

$$(\omega - c^2 k^2) \vec{E}_1 = j \underline{\omega} \text{ mono } \vec{v}_{e1} \rightarrow ④$$

\downarrow

$$= \hat{i} \vec{E}_{1x} + \hat{j} \vec{E}_{1y}$$

$$\hat{i} v_{e1x} + \hat{j} v_{e1y}$$

Separating ④ into x and y components.

$$(\omega^2 - c^2 k^2) E_{1x} = j \frac{\omega n_0 e}{\epsilon_0} v_{e1x} \rightarrow ⑤a$$

$$\& (\omega^2 - c^2 k^2) E_{1y} = j \frac{\omega n_0 e}{\epsilon_0} v_{e1y} \rightarrow ⑤b$$

Using the procedure of linearization as in case- ⑥ ,
the eqn of motion for the $e^- \rightarrow$

$$m_e \frac{d \vec{v}_{e1}}{dt} = -e \left[\vec{E}_1 + \vec{v}_{e1} \times \vec{B}_0 \right]$$

$$\Rightarrow m_e (-j \omega) (\hat{i} v_{e1x} + \hat{j} v_{e1y}) = -e \left[(\hat{i} \vec{E}_{1x} + \hat{j} \vec{E}_{1y}) + (\hat{i} v_{e1x} + \hat{j} v_{e1y}) \times \hat{k} \vec{B}_0 \right]$$

$$\Rightarrow +j m_e \omega (\hat{i} v_{e1x} + \hat{j} v_{e1y}) = +e \left[\hat{i} \vec{E}_{1x} + \hat{j} \vec{E}_{1y} - j \vec{B}_0 v_{e1x} + \hat{i} \vec{B}_0 v_{e1y} \right]$$

$$= e \left[\hat{i} (\vec{E}_{1x} + \vec{B}_0 v_{e1y}) + \hat{j} (\vec{E}_{1y} - \vec{B}_0 v_{e1x}) \right]$$

Separating the above eqn into x & y components -

$$j m_e \omega \underline{v_{e1x}} = e (\underline{E}_{1x} + \vec{B}_0 \underline{v_{e1y}})$$

$$j m_e \omega \underline{v_{e1y}} = e (\underline{E}_{1y} - \vec{B}_0 \underline{v_{e1x}})$$

$$\therefore v_{e1x} = -j \frac{e}{m_e \omega} (B_{1x} + B_0 v_{e1y}) \rightarrow (6a)$$

$$\& v_{e1y} = -j \frac{e}{m_e \omega} (E_{1y} - B_0 v_{e1x}) \rightarrow (6b)$$

Putting (6b) in (6a) →

$$v_{e1x} = \left(-j \frac{e}{m_e \omega} \right) \left[E_{1x} + B_0 \left(-j \frac{e}{m_e \omega} \right) \left(E_{1y} - B_0 v_{e1x} \right) \right]$$

$$\Rightarrow v_{e1x} \left[1 + \left(-j \frac{e}{m_e \omega} \right)^2 B_0^2 \right] = \left(-j \frac{e}{m_e \omega} \right) E_{1x} + \left(-j \frac{e}{m_e \omega} \right)^2 B_0 E_{1y}$$

$$\Rightarrow v_{e1x} \left[1 - \frac{\omega_c^2}{N^2} \right] = \frac{e}{m_e \omega} \left[-j E_{1x} - \frac{\omega_c^2}{\omega} E_{1y} \right]$$

(When $\omega_c = \frac{e B_0}{m_e \omega} \rightarrow$ cyclotron frequency)

$$\therefore v_{e1x} = \frac{e}{m_e \omega} \left[-j E_{1x} - \frac{\omega_c^2 E_{1y}}{N \cdot \omega} \right] \left(1 - \frac{\omega_c^2}{N^2} \right)^{-1} \rightarrow (7a)$$

Similarly putting (6a) in (6b) →

$$\text{H.W. } v_{e1y} = \frac{e}{m_e \omega} \left[-j E_{1y} + \frac{\omega_c^2}{\omega} E_{1x} \right] \left(1 - \frac{\omega_c^2}{N^2} \right)^{-1} \rightarrow (7b)$$

Now putting (7a) in (5a) →

$$(N^2 - \omega^2 k^2) E_{1x} = j \frac{\mu_0 \rho e}{\epsilon_0} \times \frac{e}{m_e \omega} \left[-j E_{1x} - \frac{\omega_c^2}{N} E_{1y} \right] \times \left(1 - \frac{\omega_c^2}{N^2} \right)^{-1}$$

$$= N_p^2 \left[E_{1x} - j \frac{\omega_c^2}{N} E_{1y} \right] \left(1 - \frac{\omega_c^2}{N^2} \right)^{-1}$$

$$\Rightarrow \left[(N^2 - \omega^2 k^2) \left(1 - \frac{\omega_c^2}{N^2} \right) - N_p^2 \right] E_{1x} + \left(j \frac{\omega_c^2 N_p^2}{N} \right) E_{1y} = 0.$$

(A) (B) → (8a)

Putting (7b) in (5b) \rightarrow (20)

$$\begin{aligned} (\omega^2 - c^2 k^2) E_{1y} &= j \frac{\mu_0 e}{\epsilon_0} \left(\frac{e}{m_e} \right) \left[-j E_{1y} + \frac{\omega_L}{\omega} E_{1x} \right] \left(1 - \frac{\omega_L^2}{\omega^2} \right)^{-1} \\ &= j \omega_p^2 \left[-j E_{1y} + \frac{\omega_L}{\omega} E_{1x} \right] \left(1 - \frac{\omega_L^2}{\omega^2} \right)^{-1} \\ &= \omega_p^2 \left[E_{1y} + j \frac{\omega_L}{\omega} E_{1x} \right] \left(1 - \frac{\omega_L^2}{\omega^2} \right)^{-1} \end{aligned}$$

$$\Rightarrow \underbrace{\left[\left(\omega^2 - c^2 k^2 \right) \left(1 - \frac{\omega_L^2}{\omega^2} \right) - \omega_p^2 \right] E_{1y}}_{(D)} + \underbrace{\left(-j \frac{\omega_L \omega_p^2}{\omega} \right) E_{1x}}_{(C)} = 0 \rightarrow (8b)$$

(8a) & (8b) \rightarrow

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E_{1x} \\ E_{1y} \end{pmatrix} = 0. \quad \rightarrow (9)$$

\therefore (8a) & (8b) are two simultaneous eqns for E_{1x} & E_{1y} , they will be compatible only if the determinant of coefficients vanishes, i.e.

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0. \quad \rightarrow (10).$$

$$\Rightarrow AD - BC = 0.$$

$$\Rightarrow \left[\left(\omega^2 - c^2 k^2 \right) \left(1 - \frac{\omega_L^2}{\omega^2} \right) - \omega_p^2 \right]^2 = \left(j \frac{\omega_L \omega_p^2}{\omega} \right) \left(-j \frac{\omega_L \omega_p^2}{\omega} \right)$$

$$\Rightarrow \left(\omega^2 - c^2 k^2 \right) \left(1 - \frac{\omega_L^2}{\omega^2} \right) - \omega_p^2 = \pm \frac{\omega_L \omega_p^2}{\omega}$$

$$\Rightarrow \omega^2 - c^2 k^2 = \omega_p^2 \left[1 \pm \frac{\omega_L}{\omega} \right] \left(1 - \frac{\omega_L^2}{\omega^2} \right)^{-1}$$

$$= \frac{\omega_p^2 \left(1 \pm \frac{\omega_L}{\omega} \right)}{\left(1 + \frac{\omega_L}{\omega} \right) \left(1 - \frac{\omega_L}{\omega} \right)}$$

$$\therefore \omega^2 - c^2 k^2 = \frac{\omega_p^2}{1 + n_c/\omega} \quad \text{--- (11)}$$

The \pm sign indicates that there are two possible solutions to the eqns (8a) & (8b) corresponding to two different waves that can travel along \vec{B}_0 .

Dispersion relations are —

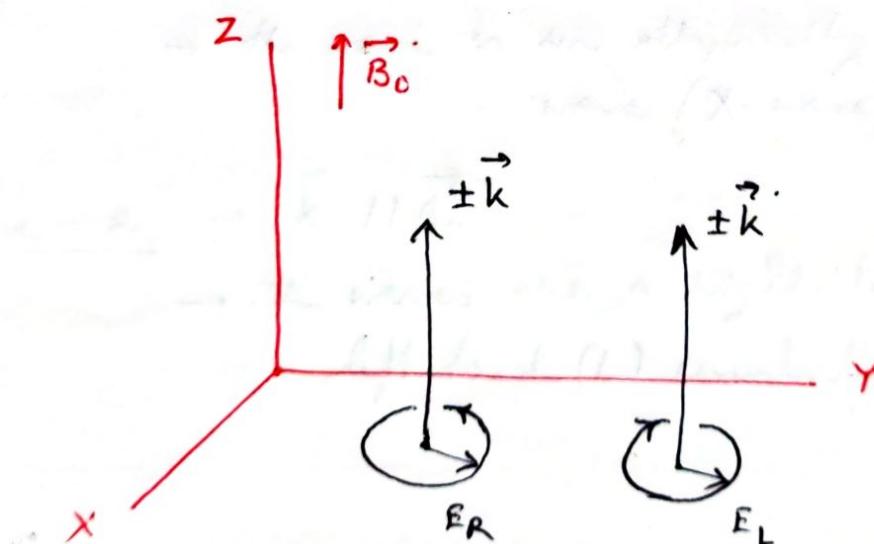
$$\boxed{\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1 + n_c/\omega}} \quad \left[\Rightarrow \frac{k^2}{v_p^2} = \frac{1}{n^2} \right] \quad \text{--- (12a)}$$

($n \rightarrow$ r.i., $v_p \rightarrow$ phase velocity)

$$\boxed{\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1 + n_c/\omega}} \quad \left[\Rightarrow \frac{c^2}{v_p^2} = n^2 \right] \quad \text{--- (12b)}$$

L-wave.

The R (right-handed) and L (left-handed) waves turn out to be circularly polarised.



Bij. geometry of R and L circularly polarised waves propagating along \vec{B}_0

(22)

The \vec{E} vector for the R-wave rotates clockwise in time as viewed along the dirⁿ of \vec{B}_0 and vice versa for L-wave. As per the dispersion relations in (12a) and (12b), the dirⁿ of rotation of \vec{E} vector is independent of the dirⁿ of \vec{k} , as eqn (12a) & (12b) depend only on k^2 . Thus, the polarisation of the wave remains the same for both the waves — wave propagating along \vec{B}_0 and the wave propagating along $-\vec{B}_0$.

next go to page 23

Summarizing —

Case - 1(a) $\rightarrow \vec{k} \perp \vec{B}_0$ and $\vec{E}_{\text{em}} \parallel \vec{B}_0$

\rightarrow the EM wave is a plane polarised wave (Ordinary wave). (O-wave)

(b) $\rightarrow \vec{k} \perp \vec{B}_0$ and $\vec{E}_{\text{em}} \perp \vec{B}_0$

\rightarrow the wave is an elliptically polarised Extraordinary wave (X-wave)

Case - 2 $\rightarrow \vec{k} \parallel \vec{B}_0$

\rightarrow the waves are a right-hand (R) and a left-hand (L) circularly polarized wave.

(23).

① Cut-offs and Resonances of R and L Waves -

Cut-offs. —

$$\text{At cutoff} - n(\text{r.i.}) \rightarrow \infty \quad | \quad n = \frac{c}{v}, \quad v = \frac{\lambda}{k}, \quad k = \frac{2\pi}{\lambda}$$

and $k = 0$.

∴ cut-off frequency for R-wave can be obtained by putting $k=0$ in dispersion eqn (12a) for R-wave.

$$\omega^2 - n_c \omega - \omega_p^2 = 0$$

Hence, cutoff for R-wave

$$\omega_R = \frac{1}{2} \left[n_c + \sqrt{n_c^2 + 4\omega_p^2} \right] \quad \rightarrow (13a)$$

Similarly, putting $k=0$ in (12b) for L-wave, we can obtain the cutoff frequency as -

$$\omega_L = \frac{1}{2} \left[-n_c + \sqrt{n_c^2 + 4\omega_p^2} \right] \quad \rightarrow (13b)$$

Resonances. —

At resonances, $n(\text{r.i.}) \rightarrow \infty$.
 $k \rightarrow \infty$.

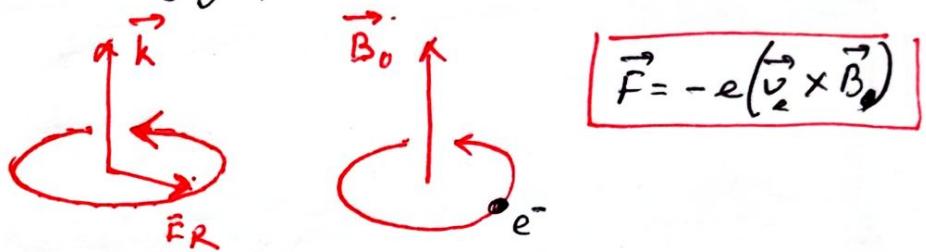
For R-wave, by dispersion relation (12a) → .

$$\frac{c^2 k^2}{\omega^2} = \frac{1 - n_c/\omega - \omega_p^2/\omega^2}{1 - n_c/\omega}$$

(24)

$$\text{At resonance} \rightarrow 1 - \frac{\omega_c}{\omega} = 0 \Rightarrow \omega = \omega_c,$$

i.e. the R-wave is in resonance with the cyclotron motion of the e^- s. The dirⁿ of motion rotation of the plane of polarisation is the same as the dirⁿ of gyration of e^- s, so the wave loses its energy in continuously accelerating the e^- s, and it cannot propagate, energy is absorbed in the medium.

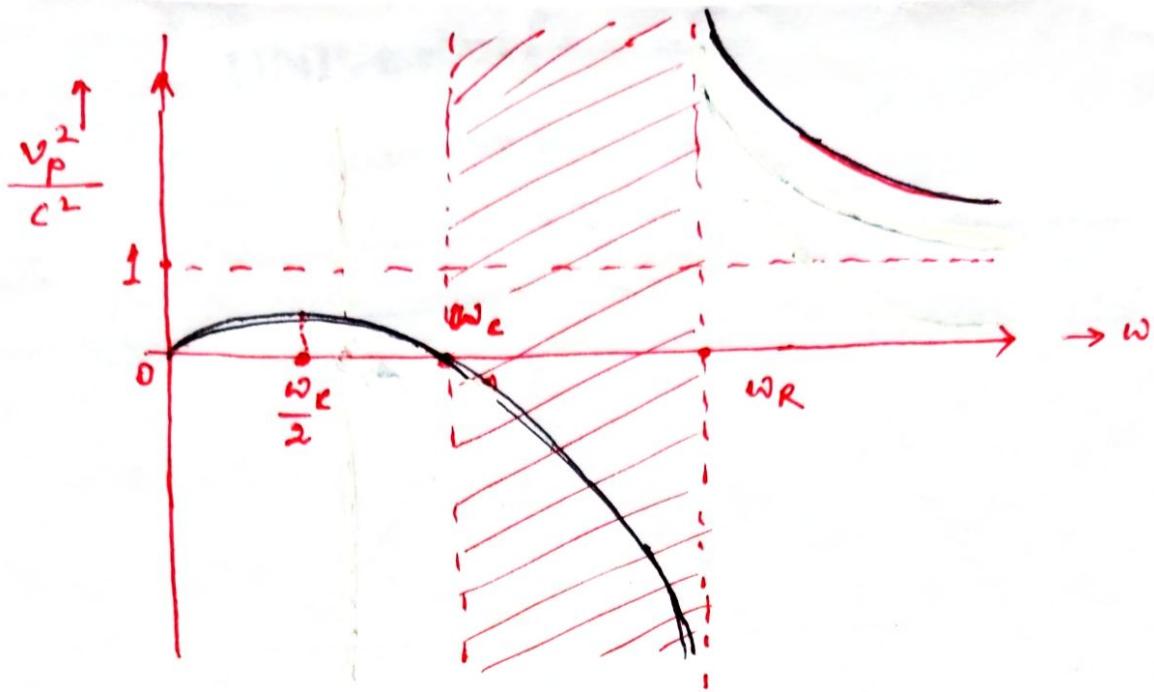


For L-wave, by dispersion relation (125), at resonance

$$\omega = -\omega_c,$$

i.e. the L-wave does not have a resonance for positive ω . But if we consider the ion motion along with the e^- motion, the game will be different and we would have a resonance at $\omega = \Omega > 0$, since the ions would now rotate with the ion gyration.

(25)



$$\text{For R-wave} \rightarrow \frac{v_p^2}{c^2} = 1 - \frac{\omega_p^2/\omega^2}{1 - \omega_c/\omega}$$

Fig-1 $\frac{v_p^2}{c^2}$ vs ω diagram for R-waves. Shaded portion of the diagram (ω_c to ω_R) is the region of nonpropagation ($\frac{v_p^2}{c^2} < 0$)

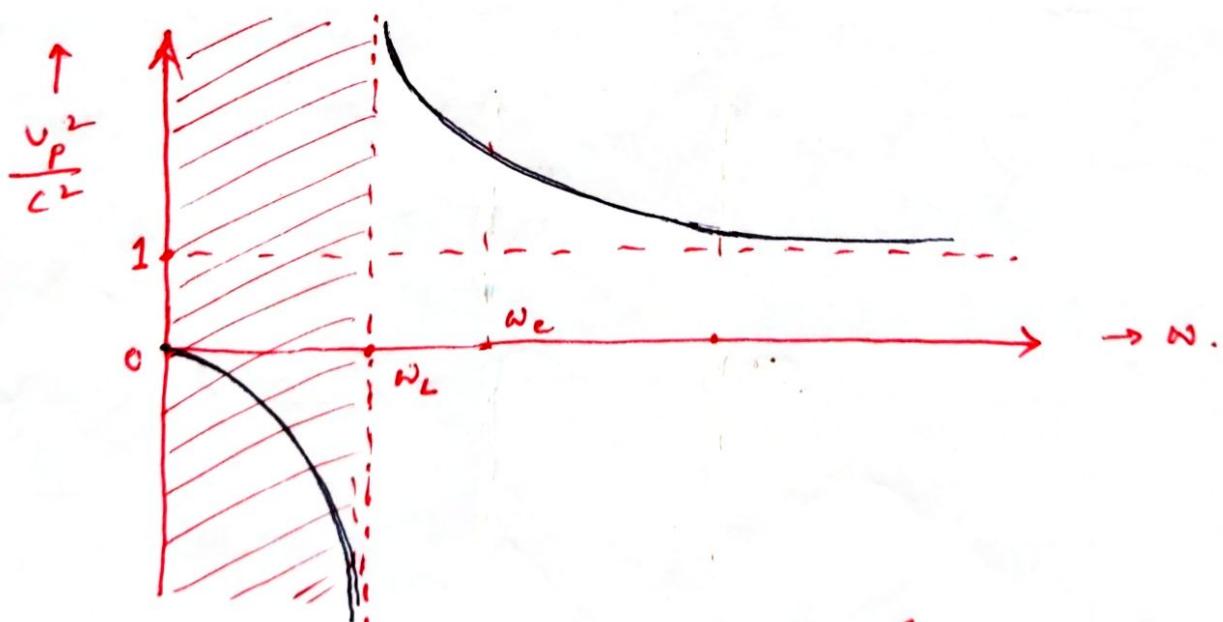


Fig-2 $\frac{v_p^2}{c^2}$ vs ω diagram for L-waves. Shaded portion of the diagram ($0 - \omega_L$) is the region of nonpropagation ($\frac{v_p^2}{c^2} < 0$).

~~Diagram of wave propagation~~

~~Medium~~ $\rightarrow X \rightarrow$

①

Potentials of e-m fields :-

* Electrostatics :- $\boxed{\vec{E} = -\nabla V}$ ① (∵ curl of div of a scalar $= 0$)
 $\nabla \times \vec{E} = 0$

V or $\phi \rightarrow$ scalar potential of \vec{E} field (stationary)

Magnetostatics :-

$$\boxed{\nabla \cdot \vec{B} = 0} \rightarrow \boxed{\vec{B} = \nabla \times \vec{A}} \quad ②$$

(∵ div curl of any vector $= 0$)

$\vec{A} \rightarrow$ vector potential of \vec{B} field (stationary).

In Electrodynamics :-

* Potentials of e-m fields when \vec{E} & \vec{B} fields are time-varying.

Maxwell's eqns -

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad ③a$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad ③b$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad ③c$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad ③d$$

(3b) All fields in electrodynamics. So, for egno.

$$\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \rightarrow ②.$$

$\vec{A} \rightarrow$ vector potential.

$$(3c) \rightarrow \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{E} = - \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{A}) = - \left(\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} \right)$$

$$\Rightarrow \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0.$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = - \vec{\nabla} \phi \rightarrow \boxed{\vec{E} = - \vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}} \rightarrow ④.$$

Once ϕ & \vec{A} are determined, \vec{B} & \vec{E} could be found in electrodynamics.

Gauge Transformations -

(2)

* Eqn (2) ^{completely} doesn't define \vec{A} for a particular magnetic field \vec{B} , because by adding the gradient of any arbitrary scalar ψ to the vector for \vec{A} , the same magnetic field \vec{B} is reproduced.

$$\vec{B} = \vec{\nabla} \times \vec{A} \longrightarrow \vec{B} = \vec{\nabla} \times (\vec{A} + \vec{\nabla} \psi)$$

$$\vec{A} \longrightarrow \boxed{\vec{A}' = \vec{A} + \vec{\nabla} \psi} \quad \text{--- (5)}$$

Though the new \vec{A}' vector ~~pot.~~ reproduces original \vec{B} field, but it is unable to reproduce the original \vec{E} field.

$$\begin{aligned} (4) \rightarrow \vec{E} &= -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} \xrightarrow{\vec{A} \rightarrow \vec{A}'} \vec{E}' = -\vec{\nabla} \phi - \frac{\partial \vec{A}'}{\partial t} \\ &= -\vec{\nabla} \phi - \frac{\partial}{\partial t} (\vec{A} + \vec{\nabla} \psi) \\ &= -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \left(\frac{\partial \psi}{\partial t} \right) \\ \vec{E}' &\neq -\vec{\nabla} \left(\phi + \frac{\partial \psi}{\partial t} \right) - \frac{\partial \vec{A}}{\partial t} \neq \vec{E}. \end{aligned}$$

The original \vec{E} field will restore, only when

$$\phi \longrightarrow \boxed{\phi' = \phi - \frac{\partial \psi}{\partial t}} \quad \text{--- (6)}$$

Thus, when \vec{A} changes to \vec{A}' by eqn (5) without affecting \vec{B} field, the scalar potential ϕ must be simultaneously transformed to ϕ' by eqn (6), so that the addition of $\vec{\nabla} \psi$ to \vec{A} doesn't affect (charge) \vec{E} along with \vec{B} field.

(3)

It can be verified by substituting \vec{A} and ϕ by \vec{A}' and ϕ' in eqn. ② for \vec{B} and in eqn. ④ for \vec{E} .

$$\vec{B} = \vec{\nabla} \times \vec{A} \longrightarrow \vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla} \phi) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla} \phi \\ = \vec{\nabla} \times \vec{A} + \vec{0}. \\ \therefore \vec{B}' = \vec{B} \quad \checkmark$$

$$\vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} \rightarrow \vec{E}' = -\vec{\nabla} \phi' - \frac{\partial \vec{A}'}{\partial t} \\ = -\vec{\nabla} \left(\phi - \frac{\partial \phi}{\partial t} \right) - \frac{\partial}{\partial t} (\vec{A} + \vec{\nabla} \phi) \\ = -\vec{\nabla} \phi + \cancel{\vec{\nabla} \left(\frac{\partial \phi}{\partial t} \right)} - \cancel{\frac{\partial \vec{A}}{\partial t}} - \cancel{\frac{\partial \vec{\nabla} \phi}{\partial t}}. \\ = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} \quad \vec{\nabla} \frac{\partial \phi}{\partial t} = \frac{\partial \vec{\nabla} \phi}{\partial t}. \\ \vec{E}' = \vec{E} \quad \checkmark$$

The meaning of ⑤ & ⑥ is that -

Any physical law which is expressed in terms of electromagnetic potentials ϕ & \vec{A} remains unaffected, if ϕ & \vec{A} are transformed by the following rules -

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla} \phi \quad \text{by ⑤}$$

$$\phi \longrightarrow \phi' = \phi - \frac{\partial \phi}{\partial t} \quad \text{by ⑥.}$$

Where ϕ is any scalar fn, called gauge fn.

These two transformations ⑤ & ⑥ are called gauge transformations.

Gauge Invariant -

(4)

If any physical law expressed in term of \vec{A} and ϕ remains invariant under the gauge transformations

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} \psi \quad \text{and} \quad \phi \rightarrow \phi' = \phi - \frac{\partial \psi}{\partial t},$$

where ψ is an ~~arbitrary~~ scalar fn., then that law is said to be a gauge invariant, as it is a must for any law in electrodynamics.

Coulomb gauge in Magnetostatics -

In magnetostatics -

$$\vec{D} \cdot \vec{B} = 0 \rightarrow \vec{B} = \vec{\nabla} \times \underline{\vec{A}}. \quad (2a) \quad \vec{A} \rightarrow \text{any arbitrary function}$$

To make \vec{A} more specific, a convenient condition is

$$\vec{D} \cdot \vec{A} = 0. \quad (2b).$$

The reason is that it makes the calculations easier than with any other choice. (In electrodynamics, this choice will be different.) Thus, in magnetostatics, we have adopted the word " $\vec{D} \cdot \vec{A} = 0$ ", which together with $\vec{B} = \vec{\nabla} \times \vec{A}$ has specified \vec{A} .

Lorentz Gauge in Electrodynamics -

In electrodynamics, in order to specify \vec{A} , we have

to impose an additional condition on \vec{A} , so that ~~the physics~~ $\textcircled{5}$ & $\textcircled{6}$ must be consistent with the transformations in $\textcircled{5}$ & $\textcircled{6}$. To derive that additional condition, we

turn our attention to the deriving the Maxwell's eqns

$\textcircled{3a}$ & $\textcircled{3d}$.

$$\textcircled{3b} \rightarrow \vec{A}, \quad \textcircled{3d} \rightarrow \phi \rightarrow \vec{E}.$$

(5).

$$\textcircled{32} \rightarrow \vec{D} \cdot \vec{E} = \frac{q}{\epsilon_0} \rightarrow \vec{D} \cdot \left[-\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} \right] = \frac{q}{\epsilon_0}$$

$$\rightarrow \boxed{\vec{D}^2 \phi + \frac{\partial}{\partial t} (\vec{D} \cdot \vec{A}) = -\frac{q}{\epsilon_0}} \rightarrow \textcircled{7}.$$

$$\textcircled{3d} \rightarrow \vec{D} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\rightarrow \vec{D} \times (\vec{A} \times \vec{B}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} \right)$$

$$\rightarrow \vec{D} (\vec{D} \cdot \vec{A}) - \vec{D}^2 \vec{A} = \mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{D} \phi) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\rightarrow -\vec{D}^2 \vec{A} + \vec{D} (\vec{D} \cdot \vec{A}) + \underbrace{\mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{D} \phi)}_{= \vec{D} \left(\mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} \right)} + \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}$$

$$\rightarrow \boxed{\vec{D}^2 \vec{A} - \vec{D} \left[\vec{D} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} \right] - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}} \rightarrow \textcircled{8}$$

a set of
Indeed of 4 Maxwell's eqns, now we are having a set of
2 eqns in $\textcircled{7}$ & $\textcircled{8}$, but they are still coupled eqns. These
two eqns carries all of the information in Maxwell's eqns.

The \vec{B} and \vec{E} are left unchanged by the Gauge transformation $\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} \phi$ by $\textcircled{5}$

$$\& \phi \rightarrow \phi' = \phi - \frac{\partial \psi}{\partial t} \quad \text{by } \textcircled{6}.$$

The freedom implied by eqns $\textcircled{5}$ & $\textcircled{6}$ means that we can choose a set of potentials (\vec{A}, ϕ) such that the middle term in eqn $\textcircled{8}$ vanishes., i.e.

$$\boxed{\vec{D} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0.} \rightarrow \textcircled{9}.$$

which is known as the Lorentz gauge in electrodynamics.
For potentials which satisfy the Lorentz gauge condⁿ, there is arbitrariness.

(6)

John Lorentz gauge will complete the pair of eqns ⑦ & ⑧ and have two inhomogeneous wave eqns, one for ϕ and one for \vec{A} :

$$⑦ \rightarrow \vec{\nabla}^2 \phi + \frac{1}{\mu_0 \epsilon_0} \left(-\mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} \right) = -\frac{f}{\epsilon_0} \quad (10)$$

$$\Rightarrow \boxed{\vec{\nabla}^2 \phi - \mu_0 \epsilon_0 \frac{\partial^2 \phi}{\partial t^2} = -\frac{f}{\epsilon_0}} \Rightarrow \vec{\nabla}^2 \phi = -\frac{f}{\epsilon_0} \rightarrow (10).$$

$$⑧ \rightarrow \boxed{\vec{\nabla}^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}} \Rightarrow \vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \rightarrow (11).$$

$$\vec{\nabla}^2 \equiv \vec{\nabla}^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \quad - \text{d'Alembertian.}$$

The above two eqns (10) & (11), plus eqn ⑨ form a set of eqns equivalent in all respects to the Maxwell's eqns in electrodynamics.

Since ϕ is associated with the scalar quantity f in eqn (10) and vector potential \vec{A} with \vec{J} in eqn (11) and both the potentials satisfy the same eqns. The Lorentz Gauge condition introduces complete symmetry b/w the scalar and vector potentials.

For the Steady-state, the time derivatives vanish and we have -

$$⑩ \rightarrow \vec{\nabla}^2 \phi = -\frac{f}{\epsilon_0} \quad \rightarrow (12)$$

$$⑪ \rightarrow \vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \quad \rightarrow (13).$$

(7)

The electric field \vec{E} and the magnetic field \vec{B} are invariant under gauge transformation -

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla} \psi \quad \text{by (5)}$$

$$\phi \longrightarrow \phi' = \phi - \frac{\partial \psi}{\partial t}. \quad \text{by (6)}$$

The transformed potentials \vec{A}' & ϕ' must satisfy the Lorentz ^{gauge} condition involving \vec{A} and ϕ -

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0. \quad (\text{by (3)})$$

Hence, the gauge function ψ which is so far remain arbitrary will have to satisfy a certain condition.

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0.$$

$$\begin{cases} \vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} \psi \\ \phi \rightarrow \phi' = \phi - \frac{\partial \psi}{\partial t} \end{cases}$$

$$\vec{\nabla} \cdot \vec{A}' + \mu_0 \epsilon_0 \frac{\partial \phi'}{\partial t} = 0.$$

$$\Rightarrow \vec{\nabla} \cdot (\vec{A} + \vec{\nabla} \psi) + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\phi - \frac{\partial \psi}{\partial t} \right) = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} + \left(\vec{\nabla} \cdot \vec{\nabla} \psi - \mu_0 \epsilon_0 \frac{\partial^2 \psi}{\partial t^2} \right) = 0.$$

The gauge transformations satisfies Lorentz gauge condition, only when the gauge function ψ satisfies the following condition -

$$\boxed{\vec{\nabla} \cdot \vec{\nabla} \psi - \mu_0 \epsilon_0 \frac{\partial^2 \psi}{\partial t^2} = 0}$$

Lorentz gauge — Importance.

(8)

Thus, the restricted gauge transformations —

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} \phi$$

$$\phi \rightarrow \phi' = \phi - \frac{\partial \phi}{\partial t}$$

where $\vec{\nabla}^2 \phi - \mu_0 \epsilon_0 \frac{\partial^2 \phi}{\partial t^2} = 0$, preserves the Lorentz gauge condition

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0,$$

provided \vec{A} and ϕ satisfy it initially.

All potentials (ϕ, \vec{A}) in this restricted class are said to belong to the Lorentz gauge.

* Importance of Lorentz gauge (ϕ, \vec{A}).

① They satisfy the wave eqns (10) & (11) →

$$\vec{\nabla}^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\vec{\nabla}^2 \phi - \mu_0 \epsilon_0 \frac{\partial^2 \phi}{\partial t^2} = -\frac{f}{\epsilon_0}.$$

which treat ϕ and \vec{A} on equivalent ~~footings~~ footings,

② Lorentz gauge is a concept which is independent of the coordinate system chosen and so it fits naturally into the domain of general relativity.

Reflection and transmission of e-m waves at the boundary of two non-conducting media.

~~Introduction~~ Here we will discuss the behaviour of e-m waves at the boundaries of two non-conducting medium. The boundary is supposed to be planar of infinite extent, the wave incident on the boundary is plane wave system. The boundary conditions are capable of reflecting back uniform plane wave and transmitting into another plane wave at the boundary. This will also show how optics is contained within the framework of Maxwell's electrodynamics.]

PART - I

Laws of reflection and transmission (refraction) - I
 We suppose that a plane e-m wave is incident obliquely on the plane boundary (X-Y) at $z=0$ separating the two non-conducting media - Medium-1 (ϵ_1, μ_1) and Medium-2 (ϵ_2, μ_2) and Y-Z plane is the plane of incidence. We, again, suppose that there will be both reflected and transmitted (refracted) waves. The e.f. \vec{E} and m.f. \vec{H} for the incident, reflected and transmitted waves are given by -

(2)

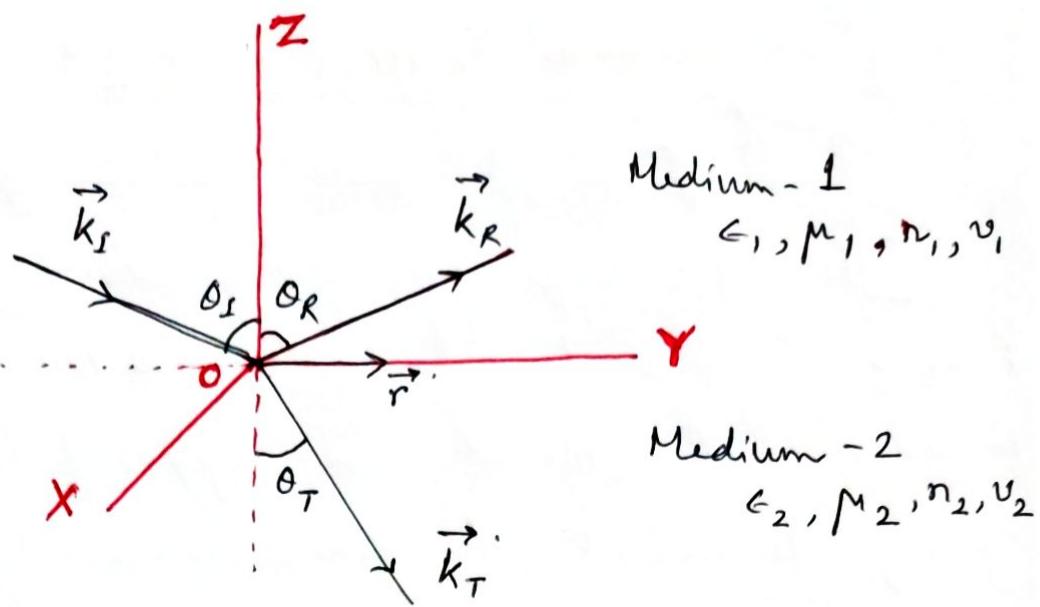


fig-1

for incident wave -

$$\vec{E}_I = E_{0I} e^{j(\vec{k}_I \cdot \vec{r} - \omega_I t)} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \left. \begin{array}{l} \therefore \vec{D} \times \vec{E}_I = -\frac{\partial \vec{B}_I}{\partial t} \\ \Rightarrow j \vec{k}_I \times \vec{E}_I = -(j \omega_I) \vec{B}_I \\ \Rightarrow H_I = \frac{\vec{k}_I \times \vec{E}_I}{\mu_I \omega_I} \end{array} \right. \quad \text{①}$$

for reflected wave -

$$\vec{E}_R = E_{0R} e^{j(\vec{k}_R \cdot \vec{r} - \omega_R t)}, \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \text{②}$$

$$\vec{H}_R = \frac{\vec{k}_R \times \vec{E}_R}{\omega_R \mu_1}$$

for transmitted wave -

$$\vec{E}_T = E_{0T} e^{j(\vec{k}_T \cdot \vec{r} - \omega_T t)}, \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \text{③}$$

$$\vec{H}_T = \frac{\vec{k}_T \times \vec{E}_T}{\omega_T \mu_2}$$

Here the subscripts I, R & T represent incident, reflected and transmitted waves respectively. The quantities E_{0I} , E_{0R} and E_{0T} are time independent scalar amplitudes (which may be complex).

(3)

These three waves on the boundary at $z=0$ on $X-Y$ plane must satisfy the following boundary conditions :

① The tangential components of \vec{E} and \vec{H} must be continuous across the boundary at all points and at all times.

② The normal components of \vec{D} and \vec{B} must be continuous.

The condition(1) demands that the frequencies are the same at the boundaries for all points and for all times for the three waves. This is possible, if

$$\omega_I = \omega_R = \omega_T (= \omega) \quad \rightarrow ④$$

$$\text{and } \vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r} \quad \rightarrow ⑤.$$

- ④ indicates that the frequency is unchanged for reflected and transmitted waves, and
- ⑤ shows that \vec{k}_I , \vec{k}_R & \vec{k}_T - all the three vectors are coplanar and therefore, incident, reflected and transmitted (refracted) waves are lying on the same plane ($Y-Z$ plane here), which is again \perp upon the plane boundary ($X-Y$ plane) separating the two media., which is one of the laws for reflection & refraction of light.

(4) If $\vec{A} \cdot \vec{r} = \vec{B} \cdot \vec{r} = \vec{C} \cdot \vec{r}$, \vec{r} is any arbitrary position vector, then \vec{A}, \vec{B} & \vec{C} must be coplanar. — prove it. (H.W.)

We suppose that the \vec{r} in eqn (5) is lying on X-Y plane (boundary plane) making angle θ_{dir} , then by eqn (5) →

$$k_I \sin \theta_I = k_R \sin \theta_R = k_T \sin \theta_T = k \sin \theta_{\text{dir}}$$

$$\text{or } k_I \sin \theta_I = k_R \sin \theta_R = k_T \sin \theta_T \rightarrow (6)$$

Since, \vec{k}_I and \vec{k}_R propagation vectors are on the same ~~dis~~ medium - I

$$\boxed{k_I = k_R}$$

and hence,

$$(6) \rightarrow k_I \sin \theta_I = k_T \sin \theta_T \rightarrow \boxed{\theta_I = \theta_T}$$

i.e. angle of incidence is equal to angle of reflection, which is ~~one of the laws for reflection~~ another law of reflection. (of two, one is already derived.)

Again, since

$$k_I \sin \theta_I = k_T \sin \theta_T$$

$$\Rightarrow \frac{\sin \theta_I}{\sin \theta_T} = \frac{k_T}{k_I} = \frac{\omega \sqrt{\epsilon_2 \mu_2}}{\omega \sqrt{\epsilon_1 \mu_1}}$$

$$= \frac{\sqrt{\epsilon_1 \mu_1}}{\sqrt{\epsilon_2 \mu_2}} = \frac{v_1}{v_2}$$

$$v = \frac{1}{\sqrt{\epsilon \mu}} = \frac{\omega}{k}$$

$$\therefore k = \omega \sqrt{\epsilon \mu}$$

$$\therefore \frac{\sin \theta_1}{\sin \theta_2} = \frac{v_2}{v_1} = \frac{n_2}{n_1} \quad | n = \frac{c}{v} \rightarrow r.i \quad (5)$$

where n_1 and n_2 — r.i. of medium 1 & 2. Thus.

$$\left[\frac{n_2}{n_1} = \frac{\sin \theta_1}{\sin \theta_2} \right] = n_1^2 \rightarrow \text{r.i. of medium 2 w.r.t. medium 1.}$$

= constant.

Which is the Snell's Law for refraction.

Reflection of e-m waves from a metal surface

Normal incidence -

We consider a dielectric-metal interface and for the sake of simplification, we treat the case of normal incidence.

We have, for incident, reflected and transmitted waves :-

$$\vec{E}_I = E_{0I} e^{j(\vec{k}_I \cdot \vec{r} - \omega_1 t)} ; H_I = \frac{\vec{k}_I \times \vec{E}_I}{\omega_1 \mu_1} \quad (1)$$

$$\vec{E}_R = E_{0R} e^{j(\vec{k}_R \cdot \vec{r} - \omega_2 t)} ; H_R = \frac{\vec{k}_R \times \vec{E}_R}{\omega_2 \mu_1} \quad (2)$$

$$\vec{E}_T = E_{0T} e^{j(\vec{k}_T \cdot \vec{r} - \omega_3 t)} ; H_T = \frac{\vec{k}_T \times \vec{E}_T}{\omega_3 \mu_2} \quad (3)$$

But $\boxed{\omega_1 = \omega_2 = \omega_3 = \omega} \rightarrow (4)$

and $\boxed{K_I = K_R} \rightarrow (5)$

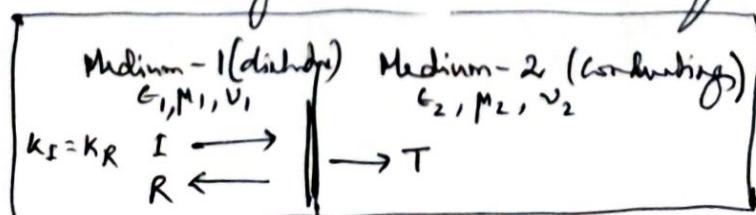
By Maxwell's eqn -

$$\vec{D} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$\vec{k} \times \vec{E} = \omega \vec{B} \rightarrow \vec{H} = \frac{\vec{k} \times \vec{E}}{\omega \mu}$$

Medium-1 (ϵ_1, μ_1, v_1) is the dielectric medium ($\sigma = 0$) where we have the incident and reflected waves, and the medium 2 is a conducting medium, where we have the transmitted wave.

The boundary conditions require that



$$(2) E_{0I} - E_{0R} = E_{0T} \rightarrow (6)$$

$$k_I(E_{0I} + E_{0R}) = k_T E_{0T} \rightarrow (7)$$

Memoise it

[The wave number (propagation constant) k_I for the incident and reflected waves in the dielectric medium-1 is real, but the propagation constant k_T for the transmitted wave in the conducting medium-2 is complex.

(The imaginary part of k_T gives rise to the exponential decay pattern in \vec{E}_T and \vec{H}_T which is responsible for attenuation in conducting medium). Since k_T is complex, E_{0R} and E_{0T} cannot both be real and hence phase shifts in the reflected and transmitted waves are other than 0 and π .]

Applying the Maxwell's eqns \rightarrow

$$\begin{cases} \nabla \cdot \vec{E} = \frac{\rho}{\epsilon} & \nabla \times \vec{B} = - \frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} = 0 & \nabla \times \vec{B} = \mu \sigma \vec{E} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \end{cases}$$

The wave equations for the electric field \vec{E} and magnetic field \vec{B} can be derived. -

$$\left(\nabla^2 - \mu \sigma \frac{\partial}{\partial t} - \mu \epsilon \frac{\partial^2}{\partial t^2} \right) \left(\frac{\vec{E}}{\vec{B}} \right) = 0. \rightarrow (8)$$

(3)

These two equations (wave eqn's) in (8) admit plane wave solutions in the medium -1 & 2 as stated in (1), (2), & (3).

Since $\sigma = 0$ for medium-1 (dielectric), the middle term in (8) also vanishes and if we put solutions from (1) & (2) in (8), we have found real values for $k_1 (= k_R)$. But if we do not mix solution from (3) for conducting medium-2 ($\sigma \neq 0$) in (8), we have -

$$k_T^2 = \frac{\mu_2 \epsilon_2 \omega^2 + j\omega \mu_2 \sigma}{\mu_2 \epsilon_2 \omega^2 [1 + j\frac{\sigma}{\epsilon_2 \omega}]} \rightarrow (9)$$

It shows that k_T is complex.

$$\text{Let } k_T = k_T^+ + jk_T^- \rightarrow (10)$$

$$\text{with } k_T^+ = \omega \sqrt{\frac{\mu_2 \epsilon_2}{2}} \left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon_2^2 \omega^2}} \right]^{\frac{1}{2}} \rightarrow (11a)$$

$$k_T^- = \omega \sqrt{\frac{\mu_2 \epsilon_2}{2}} \left[-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon_2^2 \omega^2}} \right]^{\frac{1}{2}} \rightarrow (11b)$$

$$\begin{aligned} \vec{\nabla} \cdot \vec{E}_T &= j k_T \cdot \vec{E}_T \\ \vec{\nabla} &\rightarrow j k_T \\ \vec{\nabla}^2 &\rightarrow -k_T^2 \\ &\& \frac{\partial \vec{E}_T}{\partial t} = -j \omega \vec{E}_T \\ \frac{\partial}{\partial t} &\rightarrow -j \omega. \end{aligned}$$

(The mathematics is already done in the unit - cm waves, propagation of em waves through free space, dielectric medium and conducting medium.)

Since k_T is complex, E_{0R} and E_{0T} cannot both be real and hence the phase shifts in reflected and transmitted wave are some other than 0 and π .

$$\textcircled{4} \times k_T - \textcircled{7} \Rightarrow k_T (\underbrace{E_{0I} - E_{0R}}) - k_I (\underbrace{E_{0I} + E_{0R}}) = 0.$$

$$\Rightarrow (k_T - k_I) E_{0I} - (k_T + k_I) E_{0R} = 0.$$

$$\Rightarrow \boxed{E_{0R} = \frac{k_T - k_I}{k_T + k_I} E_{0I}} \rightarrow \textcircled{12a}$$

$$\textcircled{6} \times k_I + \textcircled{7} \Rightarrow k_I (E_{0I} - E_{0R}) + k_I (E_{0I} + E_{0R})$$

$$= k_I E_{0T} + k_T E_{0T}$$

$$\Rightarrow 2k_I E_{0I} = (k_I + k_T) E_{0T}$$

$$\Rightarrow \boxed{E_{0T} = \frac{2k_I}{k_T + k_I} E_{0I}} \rightarrow \textcircled{12b}$$

$\therefore v_1 = \frac{1}{\sqrt{\epsilon_1 \mu_1}}$ → speed of the e.m. (incident + reflected) wave
in medium - 1

$$\text{and } v_1 = \frac{\omega_1}{k_I} = \frac{\omega}{k_I}$$

$$\therefore \boxed{k_I = \omega \sqrt{\epsilon_1 \mu_1}} \rightarrow \textcircled{13}.$$

Putting the values of propagation constants k_I & k_T
from $\textcircled{13}$ and $\textcircled{9}$ in $\textcircled{12a}$ and $\textcircled{12b}$, we have -

$$E_{0R} = \frac{\sqrt{\mu_2 \epsilon_2 \omega^2} \left[1 + i \frac{\sigma}{\epsilon_2 \omega} \right]^{\frac{1}{2}} - \omega \sqrt{\epsilon_1 \mu_1}}{\sqrt{\mu_2 \epsilon_2 \omega^2} \left[1 + i \frac{\sigma}{\epsilon_2 \omega} \right]^{\frac{1}{2}} + \omega \sqrt{\epsilon_1 \mu_1}} E_{0I} \rightarrow \textcircled{14a}$$

$$\& E_{0T} = \frac{2 \omega \sqrt{\epsilon_1 \mu_1}}{\sqrt{\mu_2 \epsilon_2 \omega^2} \left[1 + i \frac{\sigma}{\epsilon_2 \omega} \right]^{\frac{1}{2}} + \omega \sqrt{\epsilon_1 \mu_1}} E_{0I} \rightarrow \textcircled{14b}$$

(5)

(a) For perfect conductor (medium - 2), $\sigma \rightarrow \infty$

$$\boxed{\sqrt{\mu_2 \epsilon_2 / \omega^2} \left(1 + j \frac{\sigma}{\epsilon_2 \omega} \right)^{1/2} / \sqrt{\omega \mu_2 \epsilon_2}} \quad \boxed{E_{0R} = E_{0I}} \quad (15a)$$

$$\text{and} \quad \boxed{E_{0T} = 0} \quad (15b)$$

In this case the wave is totally reflected back from the surface of a perfect conductor.

(b) For a very good conductor (medium - 2), $\frac{\sigma}{\epsilon_2 \omega} \gg 1$

(i.e. $\sigma \gg \epsilon_2 \omega$, but not infinity).

(10), (11a), (11b) \rightarrow

$$k_T = k_T^+ + j k_T^- \\ = \omega \sqrt{\frac{\mu_2 \epsilon_2}{2}} \left\{ \left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon_2^2 \omega^2}} \right]^{1/2} + j \left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon_2^2 \omega^2}} \right]^{1/2} \right\}$$

$$= \omega \sqrt{\frac{\mu_2 \epsilon_2}{2}} \left\{ \left[1 + \frac{\sigma}{\epsilon_2 \omega} \right]^{1/2} + j \left[1 + \frac{\sigma}{\epsilon_2 \omega} \right]^{1/2} \right\}$$

$$\left(\because \sigma \gg \epsilon_2 \omega \rightarrow 1 + \frac{\sigma^2}{\epsilon_2^2 \omega^2} \approx \frac{\sigma^2}{\epsilon_2^2 \omega^2} \right)$$

$$= \omega \sqrt{\frac{\mu_2 \epsilon_2}{2}} \left\{ \sqrt{\frac{\sigma}{\epsilon_2 \omega}} + j \sqrt{\frac{\sigma}{\epsilon_2 \omega}} \right\}$$

$$= \sqrt{\frac{\omega \mu_2 \sigma}{2}} (1+j)$$

$$\therefore \boxed{k_T = \frac{1+j}{\delta}} \rightarrow (16) \quad \delta = \sqrt{\frac{2}{\omega \mu_2 \sigma}} \rightarrow \text{Skin depth}$$

(For skin depth - see the topic - propagation of e-m through conducting medium)

(6)

$$(12a) \Rightarrow E_{OR} = \frac{\frac{1+j}{\delta} - \omega \sqrt{\epsilon_1 \mu_1}}{\frac{1+j}{\delta} + \omega \sqrt{\epsilon_1 \mu_1}} E_{OI}$$

$$= \frac{\left(\frac{1}{\delta} - \omega \sqrt{\epsilon_1 \mu_1}\right) + \frac{j}{\delta}}{\left(\frac{1}{\delta} + \omega \sqrt{\epsilon_1 \mu_1}\right) + \frac{j}{\delta}}$$

 E_{OI}

→ (17)

$$\begin{aligned} |a+jb|^2 &= (a+jb)(a+jb)^* \\ &= a^2 + b^2 \end{aligned}$$

$$(12b) E_{OT} = \frac{2 \omega \sqrt{\epsilon_1 \mu_1}}{\frac{1+j}{\delta} + \omega \sqrt{\epsilon_1 \mu_1}} E_{OI}$$

$$= \frac{2 \omega \sqrt{\epsilon_1 \mu_1}}{\left(\frac{1}{\delta} + \omega \sqrt{\epsilon_1 \mu_1}\right) + \frac{j}{\delta}} E_{OI} \rightarrow (18)$$

Therefore reflection coefficient R for good conductor -

$$R = \left| \frac{E_{OR}}{E_{OI}} \right|^2 = \frac{\left(\frac{1}{\delta} - \omega \sqrt{\epsilon_1 \mu_1}\right)^2 + \frac{1}{\delta^2}}{\left(\frac{1}{\delta} + \omega \sqrt{\epsilon_1 \mu_1}\right)^2 + \frac{1}{\delta^2}}$$

$$= \frac{(1 - \omega \sqrt{\epsilon_1 \mu_1} \delta)^2 + 1}{(1 + \omega \sqrt{\epsilon_1 \mu_1} \delta)^2 + 1}$$

$$\delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$$

$$\omega \sqrt{\epsilon_1 \mu_1} \delta = \omega \sqrt{\epsilon_1 \mu_1} \times \sqrt{\frac{2}{\omega \mu_0 \sigma}}$$

$$= \sqrt{\frac{2 \omega \epsilon_1 \mu_1}{\mu_0 \sigma}}$$

$\therefore \frac{\sigma}{\epsilon_2 \omega} \gg 1 \rightarrow [\omega \sqrt{\epsilon_1 \mu_1} \delta \ll 1]$, and hence

$$R = \frac{2 + (\omega \sqrt{\epsilon_1 \mu_1} \delta)^2 - 2 \omega \sqrt{\epsilon_1 \mu_1} \delta}{2 + (\omega \sqrt{\epsilon_1 \mu_1} \delta)^2 + 2 \omega \sqrt{\epsilon_1 \mu_1} \delta}$$

→ (19)

(7)

$$= (1 - \omega\sqrt{\epsilon_1\mu_1}\delta)(1 + \omega\sqrt{\epsilon_1\mu_1}\delta)^{-1}$$

(neglecting squared term, as very very small) -

$$= (1 - \omega\sqrt{\epsilon_1\mu_1}\delta)(1 - \omega\sqrt{\epsilon_1\mu_1}\delta + \dots).$$

$$= (1 - \omega\sqrt{\epsilon_1\mu_1}\delta)^2 \quad (\text{neglecting the higher order terms in the series})$$

$$\therefore R = 1 - 2\omega\sqrt{\epsilon_1\mu_1}\delta$$

$$= 1 - 2\omega\sqrt{\frac{\epsilon_1 M_1}{M_2}} \times \sqrt{\frac{2}{\omega\mu_2\sigma}}$$

$$R = 1 - 2\sqrt{\frac{2\omega\epsilon_1 M_1}{\mu_2\sigma}}$$

for non-magnetic material $M_1 \sim M_2$

$$\therefore \boxed{R = 1 - 2\sqrt{\frac{2\omega\epsilon_1}{\sigma}}} \approx 1 \rightarrow 20a$$

Transmission coefficient for good conductor:-

$$\therefore R + T = 1 \Rightarrow \boxed{T = 1 - R = 2\sqrt{\frac{2\omega\epsilon_1}{\sigma}} \approx 0} \rightarrow 20b$$

(24)

Brewster's Angle / Polarising Angle -

Fresnel's eqn's for oblique incidence, when the electric field vector (\vec{E}) is \perp^r to the plane of incidence

is given by

$$\left(\frac{E_{0R}}{E_{0I}} \right)_{\perp} = \frac{\sin(\theta_T - \theta_I)}{\sin(\theta_T + \theta_I)} \quad \text{by eqn } (126)$$

and when \vec{E} is parallel to the plane of incidence

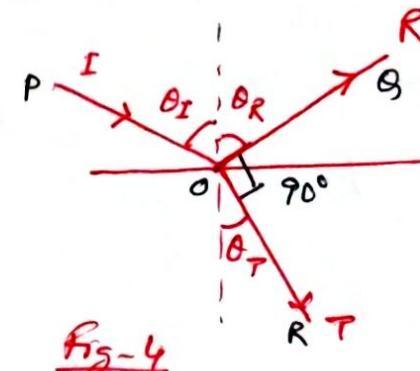
$$\left(\frac{E_{0R}}{E_{0I}} \right)_{||} = \frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)} \quad \text{by eqn } (196)$$

Let us imagine a situation where the reflected and transmitted waves are at right angle, i.e.

$$[R \ominus T = \frac{\pi}{2} \leftarrow (\text{Fig-4})]$$

$$\Rightarrow \pi - (\theta_R + \theta_T) = \frac{\pi}{2}$$

$$\Rightarrow \theta_I + \theta_T = \frac{\pi}{2} \quad \therefore \theta_I = \theta_R$$



In such a situation -

$$\left(\frac{E_{0R}}{E_{0I}} \right)_{||} = 0, \text{ as } \tan \frac{\pi}{2} \rightarrow \infty \quad (\text{by } 19b)$$

but $\left(\frac{E_{0R}}{E_{0I}} \right)_{\perp} \rightarrow \text{finite} \quad (\text{Other Non zero})$.

and so, the reflection coefficients -

$$R_{||} = \left| \frac{E_{0R}}{E_{0I}} \right|^2 = 0 \quad (\text{by eqn } 21a)$$

$$\text{and } R_{\perp} = \left| \frac{E_{0R}}{E_{0I}} \right|^2 \neq 0.$$

The physical meaning of the above two eqn's is that if we consider a general arbitrary case for the e. f. \vec{E} in the incident wave and resolve it into two rectangular components - (1) one is \perp^r to the plane of incidence (as in case-I), and (2) the other is \parallel to the plane of incidence (as in case-II) and also suppose that the two waves-reflected and transmitted are at right angle, the reflected wave has no component of E at all on the plane of incidence (since $R_{||} = 0$), but it has the component of E which is \perp^r to the plane of incidence. Thus, (since $R_{\perp} \neq 0$)

(26)

the reflected wave is perfectly plane polarised with $\vec{E} \perp^{\circ}$ to the plane of incidence. This is all about Brewster's law and the angle of incidence is called the angle of polarisation or Brewster's angle (θ_B).

The law is - If the incident wave is unpolarised and it is incident at Brewster's angle, the only component of e.f. vector \vec{E} polarised \perp° to the plane of incidence will be reflected and hence, the reflected wave is perfectly plane polarised \perp° to the plane of incidence.

By Snell's law -

$$\frac{n_2}{n_1} = \frac{\sin \theta_I}{\sin \theta_T}$$

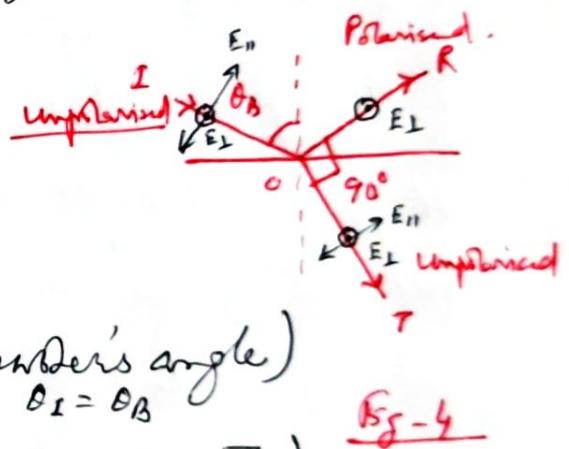
$$= \frac{\sin \theta_B}{\sin \theta_T} \quad (\text{at Brewster's angle})$$

$$= \frac{\sin \theta_B}{\sin \left(\frac{\pi}{2} - \theta_B \right)} \quad \left(\because \theta_I + \theta_T = \frac{\pi}{2} \right)$$

$$= \frac{\sin \theta_B}{\cos \theta_B}$$

$$\therefore \frac{n_2}{n_1} = \tan \theta_B \rightarrow \boxed{\theta_B = \tan^{-1} \frac{n_2}{n_1}}$$

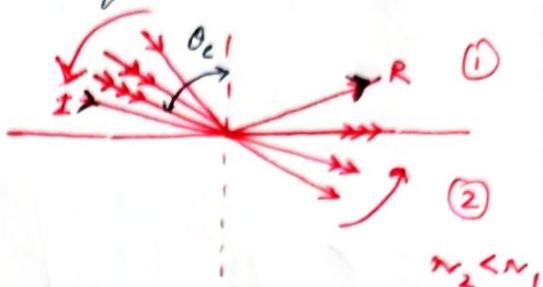
For air-glass interface $\frac{n_2}{n_1} = 1.5$ ($n_1 = 1$, $n_2 = 1.5$) $\rightarrow \theta_B \approx 56^\circ$.



Total internal reflection -

Let us consider a situation in which ^{an} ~~an~~ wave is incident from a medium of higher r. i. on the surface of a medium of lower refractive index, i.e. $n_2 < n_1$. By Snell's law -

$$\frac{n_2}{n_1} = \frac{\sin \theta_T}{\sin \theta_I}$$



and since, $n_2 < n_1 \rightarrow \theta_T < \theta_I$

Fig-5

i.e. the transmitted wave is deviated away from the normal and if we increase the angle of incidence θ_I gradually, θ_T also increases and for a particular value of θ_I , θ_T becomes $\frac{\pi}{2}$. The value of θ_I for which $\theta_T = \frac{\pi}{2}$ is called critical angle (θ_c). Applying it in Snell's law

$$\frac{n_2}{n_1} = \frac{\sin \theta_c}{\sin \frac{\pi}{2}} \Rightarrow \sin \theta_c = \frac{n_2}{n_1}$$

for air-glass.
 $\sin \theta_c = \frac{2}{3}$ $n_1
 $\theta_c \approx 42^\circ$ $n_2 \rightarrow \text{air}$$

Now, let us ask the question - what will happen if θ_I is increased further beyond θ_c , i.e. $\theta_I > \theta_c$?

Following the above two eqns -

$$\sin \theta_c = \frac{\sin \theta_I}{\sin \theta_T} \Rightarrow \sin \theta_T = \frac{\sin \theta_I}{\sin \theta_c}$$

$$\therefore \boxed{\cos \theta_T = \sqrt{1 - \frac{\sin^2 \theta_I}{\sin^2 \theta_c}}}$$

If $\theta_I > \theta_c \rightarrow \sin \theta_I > \sin \theta_c \rightarrow \cos \theta_T$ becomes imaginary

$$\therefore \cos \theta_T = i \sqrt{\frac{\sin^2 \theta_I}{\sin^2 \theta_c} - 1} = i \mathcal{Q}$$

$$\text{when } \boxed{\mathcal{Q} = \sqrt{\frac{\sin^2 \theta_I}{\sin^2 \theta_c} - 1}} \rightarrow \text{Real when } \theta_I > \theta_c.$$

(28) Now let us calculate the coefficients of reflection R_{\parallel} & R_{\perp} .

Case-(a) When \vec{E} is \perp^r to the plane of incidence -

$$(12a) \rightarrow \frac{E_{OR}}{E_{OI}} = \frac{\cos \theta_1 - \frac{n_2}{n_1} \cos \theta_r}{\cos \theta_1 + \frac{n_2}{n_1} \cos \theta_r} = \frac{\cos \theta_1 - \frac{n_2}{n_1} (j\phi)}{\cos \theta_1 + \frac{n_2}{n_1} (j\phi)}$$

$$\therefore (14a) \Rightarrow R_{\perp} = \left| \frac{E_{OR}}{E_{OI}} \right|^2 = | \quad |^2 = 1 \quad | \because A^*A = |A|^2$$

or $|E_{OR}| = |E_{OI}|$

Case-(b) When \vec{E} is $\parallel d$ to the plane of incidence -

$$(19a) \rightarrow \frac{E_{OR}}{E_{OI}} = \frac{\frac{n_2}{n_1} \cos \theta_1 - \cos \theta_r}{\frac{n_2}{n_1} \cos \theta_1 + \cos \theta_r} = \frac{\frac{n_2}{n_1} \cos \theta_1 - j\phi}{\frac{n_2}{n_1} \cos \theta_1 + j\phi}.$$

$$(21a) \rightarrow R_{\parallel} = \left| \frac{E_{OR}}{E_{OI}} \right|^2 = 1$$

$$\text{or } |E_{OR}| = |E_{OI}|.$$

Generalising the result of case-(a) & (b) \rightarrow the incident wave is totally reflected when $\theta_2 > \theta_c$. Which is known as total internal reflection. So, when $\theta_2 > \theta_c$, there will not be any refracted waves and all the energy is reflected back to the same medium. This can also be shown by computing the average energy flow across the boundary.

The average rate of energy flow into the medium 2 →

$$= \langle \vec{S}_T \rangle \cdot \hat{e}_n$$

$$= \frac{1}{2} |E_{0T}|^2 \sqrt{\frac{\epsilon_2}{\mu_2}} \hat{k}_T \cdot \hat{e}_n \quad (\text{derivation is skipped})$$

$$= \frac{1}{2} |E_{0T}|^2 \sqrt{\frac{\epsilon_2}{\mu_2}} (1 \times 1 \times \cos(\pi - \theta_T))$$

$$= \frac{1}{2} |E_{0T}|^2 \sqrt{\frac{\epsilon_2}{\mu_2}} (-j\theta) \quad (\because \cos \theta_T = j\theta)$$

= an imaginary.

The physical meaning is that no energy could be transmitted into the medium 2, if $\theta_2 > \theta_c$, i.e. $n_1 > n_2$.

Though there is no average energy flow across the surface, the field does exist on the other side of the boundary surface. Since

$$\vec{E}_T = \vec{E}_{0T} e^{j(k_T \cdot \vec{r} - wt)}$$

$$\therefore \vec{k}_T \cdot \vec{r} = k_T r \cos(\frac{\pi}{2} - \theta_T)$$

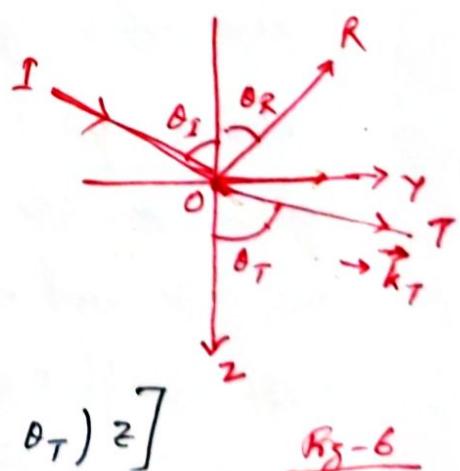
$$\vec{k}_T \cdot \vec{r} = k_T [\vec{k}_T \cdot (\hat{x} y + \hat{y} z)]$$

$$= k_T [\cos(\frac{\pi}{2} - \theta_T) y + \sin(-\theta_T) z]$$

$$= k_T (y \sin \theta_T + z \cos \theta_T)$$

$$= (y k_T \sin \theta_T + j z k_T \theta)$$

$$\therefore \vec{E}_T = \vec{E}_{0T} e^{-(k_T \theta) z} e^{j(y k_T \sin \theta_T - wt)}$$



RJ-6

(30) or $\vec{E}_T = \vec{E}_{0T} e^{-\frac{z}{k_T \delta}} e^{j(y k_T \sin \theta_T - \omega t)}$

This expression of \vec{E} of transmitted wave shows that the field does exist on the other side of the boundary even $\theta_I > \theta_c$ (when $n_1 > n_2$); but it is rapidly attenuated (exponentially attenuated). The

skin depth or penetration distance

$$\delta = \frac{1}{k_T \delta} = \frac{1}{k_T} \left(\frac{\sin^2 \theta_I}{\sin^2 \theta_c} - 1 \right)^{-\frac{1}{2}}$$

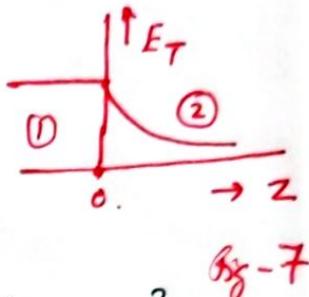


Fig-7

For glass-air combination - $\sin \theta_c = \frac{n_2}{n_1} = \frac{2}{3}$

$$\theta_c \approx 42^\circ$$

$$\text{If } \theta_2 = 45^\circ > \theta_c \rightarrow \delta = \frac{\lambda}{2\pi} \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 \times \left(\frac{3}{2}\right)^2 - 1}$$

$$\delta = 0.45\lambda$$

Thus the wave becomes negligible beyond a distance of the order of a few wavelengths.

Microscopic explanation of this attenuation - The molecular charges oscillate due to the interaction with the incident wave in the second medium and these oscillations give rise to a radiation field propagating in forward dir^n. The forward wave interferes destructively with the original wave and gives rise to a small transmission.

(31)

Again it is seen from the above discussion that when $\theta_1 > \theta_c$ and $n_2 < n_1$, there is no transport of energy across the boundaries, but there is an exponentially decaying field on the far side of the boundary (Fig-7). How is it possible? → Energy definitely flows in the 2nd medium since the component of the field is finite in that medium is finite, but it occurs in the way - ~~in~~^{During} the first half of the cycle, if the flow of energy occurs ^{into the 2nd medium} in the forward dirⁿ, during the next half of the cycle, the energy is returned back to the first medium.

Electromagnetic Waves (e-m waves) -

Wave Equations:-

Maxwell's eqn's (four eqn's) provide us with all the information that can be drawn from the classical theory of electric & magnetic fields. Maxwell eqn's combine all the theories of electrostatics and magnetostatics into a single theory - known as e-m theory. Maxwell's theory of electromagnetism predicted the existence of e-m waves travelling with speed equal to that of light.

For a homogeneous (linear) medium - ✓

$\left. \begin{array}{l} \epsilon \rightarrow \text{permittivity} \\ \mu \rightarrow \text{permeability} \\ \sigma \rightarrow \text{conductivity, (if the medium is a)} \\ \text{conducting one} \end{array} \right\}$

 And all are scalar constants, as the medium is homogeneous.

The four Maxwell's e-m (MEM) eqn's are -

$$\vec{\nabla}_0 \cdot \vec{E} = \frac{\rho}{\epsilon} \quad - (1a) \quad \left| \quad \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad \rightarrow (1c) \right.$$

$$\vec{\nabla}_0 \cdot \vec{B} = 0 \quad - (1b) \quad \left| \quad \vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \rightarrow (1d) \right.$$

and $\vec{J} = \sigma \vec{E}$ (Ohm's law - for a homogeneous conducting medium)

→ (2).

② Now, taking the curl of eqn ① →

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = - \vec{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t} \right).$$

$$\Rightarrow \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = - \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B})$$

$$\Rightarrow \vec{\nabla} \left(\frac{\rho}{\epsilon_0} \right) - \vec{\nabla}^2 \vec{E} = - \frac{\partial}{\partial t} \left(\mu_0 \vec{E} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \right) \quad \left| \text{by } \begin{matrix} 1d \\ 2, 2 \end{matrix} \right.$$

$$= - \mu_0 \frac{\partial \vec{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \vec{\nabla}^2 \vec{E} - \mu_0 \frac{\partial \vec{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{\nabla} \left(\frac{\rho}{\epsilon_0} \right).$$

Suppose $\boxed{\rho = 0}$ * (for a dielectric medium), which is again justified for conducting medium.

$$\therefore \boxed{\vec{\nabla}^2 \vec{E} - \mu_0 \frac{\partial \vec{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0.} \rightarrow ③.$$

Repeating the same exercise with the eqn ① and

and applying eqn's ② and ①b, →

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times (\mu_0 \vec{E}) + \vec{\nabla} \times \mu \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \vec{\nabla} \left(\vec{\nabla} \cdot \vec{B} \right) - \vec{\nabla}^2 \vec{B} = \mu_0 (\vec{\nabla} \times \vec{E}) + \mu \epsilon \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

$$\Rightarrow - \vec{\nabla}^2 \vec{B} = \mu_0 \left(- \frac{\partial \vec{E}}{\partial t} \right) + \mu \epsilon \frac{\partial}{\partial t} \left(- \frac{\partial \vec{E}}{\partial t} \right)$$

$$\Rightarrow \boxed{\vec{\nabla}^2 \vec{B} - \mu_0 \frac{\partial \vec{B}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0} \rightarrow ④$$

The eqn's ③ and ④ are the wave equations

for a homogeneous isotropic medium. These two differential eqn's are equivalent to the four first order Maxwell's eqn's, but they have the advantage that we have separate eqn's for E & B .

* If there is a charge density within a conducting medium, it will fall off to zero exponentially.

(3)

The charge will diffuse to the surface in an extremely short time.

With the help of a scalar fn $\psi(\vec{r}, t)$, a general wave eqnⁿ in electromagnetism could be written as -

$$\boxed{\vec{\nabla}^2 \psi - \mu \epsilon \frac{\partial \psi}{\partial t} - \mu \epsilon \frac{\partial^2 \psi}{\partial t^2} = 0} \rightarrow \textcircled{5}$$

Combining eqn's $\textcircled{3}$ & $\textcircled{4}$. -

$$\text{or } \boxed{\left(\vec{\nabla}^2 - \mu \epsilon \frac{\partial}{\partial t} - \mu \epsilon \frac{\partial^2}{\partial t^2} \right) \left(\frac{\vec{E}}{\vec{B}} \right) = 0} \rightarrow \textcircled{6}$$

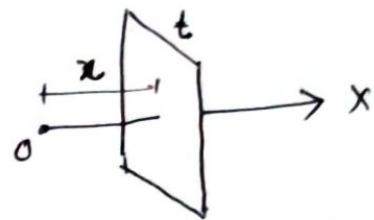
This eqn $\textcircled{6}$ shows that e-m fields can exist in absence of ~~the~~ electric charge and they can propagate through the medium with certain speed.

Solution of the wave eqnⁿ - $\textcircled{5}$ or $\textcircled{6}$. -

The simplest type of wave is a plane wave and it is a good approximation to the actual waves in many situations. Again, as ~~the~~ no boundaries are implied in our unbounded medium, plane progressive waves are the most appropriate solutions for the e-m wave eqn's.

④

Note - A plane wave is one where the surfaces of constant phase are planes normal to the dirⁿ of propagation of the wave. Such a plane is called wavefront, which advances with a certain speed, say v , in a direction normal to itself.



The solution of wave equation is considerably simplified, if we suppose that the wave is propagating along +x-direction and the field vectors \vec{E} and \vec{B} (e.f. and m.f. respectively) lie on the plane \perp^r to x-axis and they are functions of x & t , not of y and z .

$$⑥ \rightarrow \left(\frac{\partial^2}{\partial x^2} - \mu_0 \frac{\partial}{\partial t} - \mu \epsilon \frac{\partial^2}{\partial t^2} \right) \left(\frac{E(x, t)}{B(x, t)} \right) = 0. \quad \rightarrow ⑥'$$

$$⑤ \rightarrow \left(\frac{\partial^2}{\partial x^2} - \mu_0 \frac{\partial}{\partial t} - \mu \epsilon \frac{\partial^2}{\partial t^2} \right) H(x, t) = 0. \quad \rightarrow ⑤'$$

Solⁿ of ⑤ →

$$\boxed{H(x, t) = H_0 e^{-\gamma x} e^{j \omega t}} \quad (\text{trial soln}) \rightarrow ⑦$$

Amplitude damping factor → oscillation.

where $\omega \rightarrow$ angular frequency of the wave
 $\gamma \rightarrow$ decay constant.

Solution (7) must satisfy wave eqn (5'). (5)

$$\begin{aligned}\frac{\partial \psi}{\partial n} &= \frac{\partial}{\partial n} (\psi_0 e^{-\gamma x} e^{j\omega t}) \\ &= \psi_0 e^{j\omega t} \frac{\partial}{\partial x} (e^{-\gamma x}) \\ &\quad \xrightarrow{-\gamma e^{-\gamma x}}\end{aligned}$$

$$\begin{aligned}&= -\gamma \psi \quad \xrightarrow{\text{arrow}} \\ \therefore \frac{\partial^2 \psi}{\partial x^2} &= \frac{\partial}{\partial n} \left(\frac{\partial \psi}{\partial n} \right) = \frac{\partial}{\partial n} (-\gamma \psi) = -\gamma \left(\frac{\partial \psi}{\partial n} \right) = -\gamma (-\gamma \psi)\end{aligned}$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial x^2} \equiv +\gamma^2}$$

$$\begin{aligned}\frac{\partial \psi}{\partial t} &= \psi_0 e^{-\gamma x} \frac{\partial}{\partial t} (e^{j\omega t}) = \psi_0 e^{-\gamma x} (j\omega e^{j\omega t}) \\ &\quad \xrightarrow{j\omega \psi}\end{aligned}$$

$$\therefore \boxed{\frac{\partial}{\partial t} \equiv j\omega}$$

$$\text{And } \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial \psi}{\partial t} \right) = \frac{\partial}{\partial t} (j\omega \psi) = j\omega \left(\frac{\partial \psi}{\partial t} \right) = j\omega (j\omega \psi)$$

$$\therefore \boxed{\frac{\partial^2}{\partial t^2} \equiv -\omega^2} \quad (\because j^2 = -1).$$

Hence, by (5') \rightarrow

$$(\gamma^2 - j\mu\sigma\omega + \mu\epsilon\omega^2) \psi(x, t) = 0.$$

Since $\psi(x, t)$ is always non-zero, hence

$$\gamma^2 - j\mu\sigma\omega + \mu\epsilon\omega^2 = 0.$$

$$\text{or } \gamma^2 = -\mu\epsilon\omega^2 + j\mu\sigma\omega \quad \rightarrow (8)$$

Eqn (8) shows that γ is a complex quantity. So, to

⑥ evaluate γ , we suppose that -

$$\gamma = \alpha + j\beta. \quad \rightarrow \textcircled{9}.$$

$$\textcircled{9}^2 \rightarrow \gamma^2 = \alpha^2 - \beta^2 + 2j\alpha\beta \rightarrow \textcircled{10}.$$

Comparing \textcircled{8} & \textcircled{10} \rightarrow

$$\left. \begin{aligned} \alpha^2 - \beta^2 &= -\mu\epsilon\omega^2 \\ 2\alpha\beta &= \mu\sigma\omega \end{aligned} \right] \begin{matrix} \textcircled{a} \\ \textcircled{b} \end{matrix} \rightarrow \textcircled{11}$$

$$\textcircled{11a}^2 + \textcircled{11b}^2 \Rightarrow (\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2 = (\mu\epsilon\omega^2)^2 + (\mu\sigma\omega)^2$$

$$\Rightarrow (\alpha^2 + \beta^2)^2 = \mu^2\epsilon^2\omega^4 \left(1 + \frac{\sigma^2}{\epsilon^2\omega^2}\right)$$

$$\Rightarrow \alpha^2 + \beta^2 = \mu\epsilon\omega^2 \sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} \rightarrow \textcircled{11c}$$

$$\textcircled{11a} + \textcircled{11c} \Rightarrow 2\alpha^2 = \mu\epsilon\omega^2 \left[-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} \right]$$

$$\Rightarrow \left[\alpha = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} \right]^{\frac{1}{2}} \right] \rightarrow \textcircled{12a}$$

$$\textcircled{11c} - \textcircled{11a} \Rightarrow 2\beta^2 = \mu\epsilon\omega^2 \left[\sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} + 1 \right]$$

$$\Rightarrow \left[\beta = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} \right]^{\frac{1}{2}} \right] \rightarrow \textcircled{12b}$$

Now by eqn \textcircled{5} \rightarrow

$$\gamma = \omega \sqrt{\frac{\mu\epsilon}{2}} \left\{ \left[-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} \right]^{\frac{1}{2}} + j \left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\omega^2}} \right]^{\frac{1}{2}} \right\} \rightarrow \textcircled{13}$$

(7)

Thus the general solution -

$$(7) \rightarrow \psi(x, t) = \psi_0 e^{-(\alpha + j\beta)x} e^{j\omega t} \quad (\text{by } (7))$$

$$\therefore \boxed{\psi(x, t) = \psi_0 e^{-\alpha x} e^{j(\omega t - \beta x)}} \rightarrow (14)$$

Thus \vec{E} and \vec{B} at $x = x$ & at $t = t$ →

$$\boxed{\vec{E}(x, t) = \vec{E}_0 e^{-\alpha x} e^{j(\omega t - \beta x)}} \rightarrow (15)$$

$$\boxed{\vec{B}(x, t) = \vec{B}_0 e^{-\alpha x} e^{j(\omega t - \beta x)}} \rightarrow (16)$$

where α & β are given by (12a) & (12b).

Eqs (15) & (16) indicate that two oscillatory e.f. \vec{E} and m.f. \vec{B} are propagating along $+x$ -dir with attenuation ($\because \alpha \neq 0$) through the medium. The quantity α is called the absorption coefficient or attenuation constant for the e-m wave in that medium. α is the measure of attenuation of the e-m wave in the medium.

* Phase Velocity - It is the velocity of the wave with which the wavefront of the wave advances in the dir $\perp r$ to it. For a wavefront at $x = x$, at $t = t$, the phase

$$\phi = \omega t - \beta x = \text{constant.}$$

$$\text{or } \omega dt - \beta dx = 0$$

$$\text{or } \frac{dx}{dt} = \boxed{\frac{\omega}{\beta}} = v \rightarrow \text{Phase velocity of the wave}$$

(17)

$$\therefore \vec{E}\left(x + \frac{2\pi}{\beta}, t\right) = \vec{E}_0 e^{-\alpha\left(x + \frac{2\pi}{\beta}\right)} e^{j(\omega t - \beta x + 2\pi)}$$

$$= \vec{E}_0 e^{-\alpha\left(x + \frac{2\pi}{\beta}\right)} e^{j(\omega t - \beta x)} \rightarrow (18)$$

$$\left(\because e^{j2\pi} = +1 \right)$$

$$\cos 2\pi + j \sin 2\pi = +1$$

By eqns (15) and (18), the phase of electric field at $x = x$ and at $x = x + \frac{2\pi}{\beta}$ is same. So, by defn of wavelength -

$$\text{Wavelength of the wave } \lambda = \frac{2\pi}{\beta}$$

$$\text{or } \boxed{\beta = \frac{2\pi}{\lambda} = k} \rightarrow (19)$$

where k is the wave number.

$$(19) \text{ in } (17) \rightarrow v = \frac{\omega}{\beta} = \frac{\omega}{k} = \frac{2\pi v}{2\pi/\lambda}$$

$$\therefore \boxed{v = \frac{\omega}{k} = v \lambda} \rightarrow (20)$$

(19) in (15) & (16) →

$$\boxed{\vec{E}(x, t) = \vec{E}_0 e^{-\alpha x} e^{j(\omega t - kx)}} \rightarrow (21a)$$

$$\boxed{\vec{B}(x, t) = \vec{B}_0 e^{-\alpha x} e^{j(\omega t - kx)}} \rightarrow (21b)$$

Sols of the wave eqns in (6). for \vec{E} and \vec{B} , α is given by (12a)

(9)

Case-I - Propagation of e-m waves in free space or vacuum

For free space $\sigma = 0$.

$$c \rightarrow \epsilon_0 = 8.85 \times 10^{-12} F/m - \text{absolute permittivity}$$

$$\mu \rightarrow \mu_0 = 4\pi \times 10^{-7} H/m - \text{absolute permeability}$$

Wave eqns (23) become \rightarrow

$$\boxed{\vec{\nabla}^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \rightarrow (23)$$

$$\boxed{\vec{\nabla}^2 \vec{B} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0} \rightarrow (24)$$

Solⁿ of (23) \rightarrow

Considering the propagation of the wave along

+X-dirⁿ -

$$(23) \rightarrow \frac{\partial^2 \vec{E}(x, t)}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}(x, t)}{\partial t^2} = 0 \rightarrow (23')$$

$$(24) \rightarrow \frac{\partial^2 \vec{B}(x, t)}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}(x, t)}{\partial t^2} = 0 \rightarrow (24')$$

Solⁿ of (23') for e.f. \vec{E} -

$$\boxed{\vec{E}(x, t) = \vec{E}_0 e^{j(\omega t - kx)}} \rightarrow (25)$$

by eqn (15)

Solⁿ of (24') for m.f. \vec{B}

$$\boxed{\vec{B}(x, t) = \vec{B}_0 e^{j(\omega t - kx)}} \rightarrow (26)$$

by eqn (16)

Here $\alpha = 0$, \therefore the attenuation constant is zero, the e-m wave can travel through vacuum without attenuation. ($\lim \sigma = 0$ for free space, by eqn (12a), $\alpha = 0$).

(10).

Phase velocity of the e-m wave through vacuum \rightarrow .

$$v = \frac{\omega}{\beta} \quad \text{by eqn } 17.$$

and by eqn 126 and 19 \rightarrow

$$\boxed{\beta = \omega \sqrt{\mu_0 \epsilon_0}} \quad (\because \sigma = 0) \quad (27)$$

$$\therefore \boxed{v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}} = c \quad (27)$$

$$= \frac{1}{\sqrt{8.85 \times 10^{-12} \times 4\pi \times 10^{-7}}}$$

$$v = 2.99784 \times 10^8 \text{ m/s.}$$

which is the speed of e-m through vacuum or free space. It is again the same as the speed of light 'c' through free space. Since the experimentally determined speed of light through free space was found to be equal to the theoretically calculated speed of e-m waves through free space, it was concluded that light is simply a form of e-m wave.

('c' is taken from the Latin word 'celeritas', meaning is : swiftness.).

(11)

Case-II Propagation of e-m waves in dielectric medium (non conducting medium) homogeneous
for dielectric (non conducting) medium - isotropic

$$\sigma = 0$$

Wave eqn's in (6) become -

$$\boxed{\vec{D}^2 \vec{E} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \longrightarrow (28)$$

$$\boxed{\vec{D}^2 \vec{B} - \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0} \longrightarrow (29)$$

Considering the propagation of the wave along $+x$ -direction.

$$(28) \rightarrow \frac{\partial^2 \vec{E}(x,t)}{\partial x^2} - \mu \epsilon \frac{\partial^2 \vec{E}(x,t)}{\partial t^2} = 0 \rightarrow (28')$$

$$(29) \rightarrow \frac{\partial^2 \vec{B}(x,t)}{\partial x^2} - \mu \epsilon \frac{\partial^2 \vec{B}(x,t)}{\partial t^2} = 0 \rightarrow (29')$$

By eqn (15), sol'n of wave eqn (28') for e.f. $\vec{E} \rightarrow$
 $\vec{E}(x,t) = \vec{E}_0 e^{j(\omega t - kx)} \rightarrow (30)$

& by eqn (16), sol'n of wave eqn (29') for m.f. $\vec{B} \rightarrow$
 $\vec{B}(x,t) = \vec{B}_0 e^{j(\omega t - kx)} \rightarrow (31)$

Here $\alpha = 0$, $\therefore \sigma = 0$ by eqn (12a), i.e.
 the attenuation constant / absorption coefficient
 is zero. and therefore, the e-m wave can travel
 through dielectric medium (non-conducting
 medium) without attenuation.

(12)

By eqn (17), the phase velocity of the wave is -

$$v = \frac{\omega}{\beta} = \frac{\omega}{k} \quad (\because \beta = k) \text{, by eqn (19)}$$

Again, by eqn (12b) →

$$\boxed{\beta = \omega \sqrt{\mu \epsilon} = k} \quad \rightarrow \quad (32)$$

$$\therefore \boxed{v = \frac{1}{\sqrt{\mu \epsilon}}} \rightarrow \text{speed of the em wave through dielectric medium.}$$

$$\therefore c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \rightarrow \text{speed of light(em wave) through vacuum}$$

$$\therefore \boxed{v = \frac{c}{\sqrt{\mu_r \epsilon_r}}} \quad \left| \begin{array}{l} \because \mu = \text{proper} \quad \& \epsilon = \epsilon_0 \epsilon_r \\ \downarrow \qquad \qquad \qquad \downarrow \\ \text{Relative permeability} \quad \text{Relative permittivity} \end{array} \right.$$

Thus, r.i. of the dielectric medium -

$$\boxed{n = \frac{c}{v} = \sqrt{\mu_r \epsilon_r}} \quad \text{by eqn (34).} \quad \rightarrow (35)$$

For most materials (non-magnetic),

$$\mu \approx \mu_0, \text{ i.e. } \mu_r \approx 1$$

$$\therefore \boxed{n = \sqrt{\epsilon_r}} \quad \rightarrow (36)$$

$$\text{and } \boxed{v = \frac{c}{\sqrt{\epsilon_r}}} \quad \text{by eqn (34)} \quad \rightarrow (37)$$

Since ϵ_r is always greater than 1 for any dielectric medium, r.i. is greater than 1 and the e-m waves move slowly in that medium ($v < c$) compared to vacuum.

Case-III Propagation of e-m waves through (13)
homogeneous isotropic conducting medium

for conducting medium -

$$\sigma \neq 0 \quad \text{and} \quad \frac{\sigma}{\omega \epsilon} \gg 1 \quad \left. \begin{array}{l} \text{by eqn } 12a \text{ & } 12b \\ \alpha = \beta \approx \sqrt{\frac{1}{2} \omega \mu \sigma} \end{array} \right\} \rightarrow (38)$$

Wave equations are -

① for e.f. \vec{E}

$$\boxed{\nabla^2 \vec{B} - \mu \sigma \frac{\partial \vec{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \rightarrow (39)$$

② for m.f. \vec{B}

$$\boxed{\nabla^2 \vec{B} - \mu \sigma \frac{\partial \vec{B}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0} \rightarrow (40)$$

Considering the propagation of e-m wave along +X-dirn -

$$(39) \rightarrow \frac{\partial^2 \vec{E}(x,t)}{\partial x^2} - \mu \sigma \frac{\partial \vec{E}(x,t)}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{E}(x,t)}{\partial t^2} = 0 \rightarrow (39')$$

$$(40) \rightarrow \frac{\partial^2 \vec{B}(x,t)}{\partial x^2} - \mu \sigma \frac{\partial \vec{B}(x,t)}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{B}(x,t)}{\partial t^2} = 0 \rightarrow (40')$$

by eqn (15), now of (39') for e.f. \vec{E} -

$$\vec{E}(x,t) = \vec{E}_0 e^{-\left(\sqrt{\frac{1}{2} \omega \mu \sigma}\right)x} e^{j(\omega t - kx)}$$

$$\text{or } \vec{E}(x,t) = \vec{E}_0 e^{-\frac{x}{\delta}} e^{j(\omega t - kx)} \rightarrow (41)$$

and by eqn (16), now of (40') for m.f. \vec{B} -

$$\vec{B}(x,t) = \vec{B}_0 e^{-\frac{x}{\delta}} e^{j(\omega t - kx)} \rightarrow (42)$$

$$\text{where } \delta = \sqrt{\frac{2}{\omega \mu \sigma}} \rightarrow (43) \quad \delta \rightarrow \text{skin depth.}$$

(14) Since $\alpha \neq 0$, for the e-m wave propagating through a conducting medium, the wave suffers attenuation inside the medium.
 (When the e-m wave is propagating through the conducting medium, the oscillatory e.f. \vec{E} in the wave sets up currents by driving the free e's in the conducting medium and so, the wave does work in driving the current spending its energy. That results into the attenuation of the wave inside the medium.)

$$\underline{\text{Skin depth}} \quad \left[\delta = \sqrt{\frac{2}{\omega \mu_0}} = \frac{1}{\alpha} \right] \rightarrow (43).$$

The amplitude of the e-m wave decreases exponentially as it goes on penetrating a conducting medium (as indicated by the exponential decay term $e^{-x/\delta}$ in eqns (41) & (42)) and thus the wave cannot propagate without attenuation through a conducting medium. The amplitude decreases to $\frac{1}{e}$ th (surface) of its value at $x=0$, after traversing a distance $x=\delta$ into the conducting medium and this distance δ is called the skin depth.

The skin depth depends on the frequency (angular frequency ω) of the e-m wave and conductivity (σ) of the conducting medium.

Greater the attenuation, smaller is the skin depth and vice-versa (by eqn (43)).

(a) For good conductor for a given frequency, σ is high, so the attenuation is ^{rapid}~~large~~ and skin depth is very short.

(b) For dielectric medium and free space $\sigma = 0 \rightarrow \alpha = 0$ (no attenuation at all) and $\delta \rightarrow \infty$ (i.e. the wave can move to infinity)

Phase Velocity through conducting medium

$$v = \frac{\omega}{\beta} \quad (\text{by eqn (17)})$$

$$\boxed{v = \sqrt{\frac{2}{\mu \epsilon}} \left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2 \omega^2}} \right]^{-1/2}} \quad \text{by eqn (126).}$$

r.i. of conducting medium -

$$\boxed{n = \frac{c}{v} = \frac{c}{\omega \beta} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \frac{1}{\beta} \sqrt{\frac{\mu \epsilon}{2}} \left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2 \omega^2}} \right]^{1/2}}$$

$$n = \sqrt{\frac{\mu_r \epsilon_r}{2}} \left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2 \omega^2}} \right]^{1/2}$$

Wavelength :-

$$k = \frac{2\pi}{\lambda} = \beta \Rightarrow \boxed{\lambda = \frac{2\pi}{\beta}}$$

(16)

Conductor, dielectric and quasi-conductor -

Apart from the nature of the substance, the frequency of the wave also plays an important factor in determining whereas a medium behaves like conductor or dielectric or quasi-conductor. By Maxwell's eqn -

$$\vec{D} \times \vec{B} = \mu \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} = \mu \sigma \vec{E} + j \omega \mu \epsilon \vec{E}.$$

Three conditions can be derived from this eqn considering the relative magnitudes of the two current densities - ① conduction current density $|\sigma \vec{E}|$ and ② displacement current density $|j \omega \epsilon \vec{E}|$.

These three conditions are -

① $\frac{\sigma}{\omega \epsilon} \gg 1$, when the conduction current density much greater than displacement current density, the medium is called conductor.

② $\frac{\sigma}{\omega \epsilon} \ll 1$, when displacement current density term is very much greater than the conduction current density term, the medium is called dielectric.

③ $\frac{\sigma}{\omega \epsilon} \approx 1$, if both the current density terms are of same order, the medium behaves as quasi-conductor.

⑥

Reflection and transmission coefficient,

Fresnel's equations -

When an e-m wave travelling in one medium strikes the boundary between two media, there are reflected as well as refracted disturbances.

For incident wave -

$$\vec{E}_I = \vec{E}_{0I} e^{j(\vec{k}_I \cdot \vec{r} - \omega_I t)}, \quad \vec{H}_I = \frac{\vec{k}_I \times \vec{E}_I}{\omega_I \mu_1} \quad (1)$$

For reflected wave -

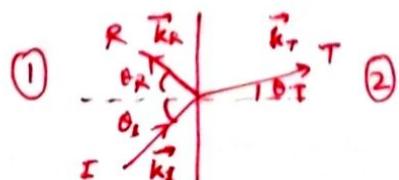
$$\vec{E}_R = \vec{E}_{0R} e^{j(\vec{k}_R \cdot \vec{r} - \omega_R t)}, \quad \vec{H}_R = \frac{\vec{k}_R \times \vec{E}_R}{\omega_R \mu_2} \quad (2)$$

For transmitted wave -

$$\vec{E}_T = \vec{E}_{0T} e^{j(\vec{k}_T \cdot \vec{r} - \omega_T t)}, \quad \vec{H}_T = \frac{\vec{k}_T \times \vec{E}_T}{\omega_T \mu_2} \quad (3)$$

Here, incident and reflected waves are lying in the medium (non-conducting) - 1 ($\epsilon_1, \mu_1, n_1, v_1$) and transmitted wave in the non-conducting medium - 2 ($\epsilon_2, \mu_2, n_2, v_2$). The plane e-m wave is incident obliquely on the plane of the boundaries separating the two media (non-conducting media) 1 & 2.

The above three waves on the boundary between the two media 1 & 2 must satisfy the following basic boundary conditions -



(7)

① The tangential components of \vec{E} and \vec{H} must be continuous across the boundaries at all points and at all times.

② The normal components of \vec{D} and \vec{B} must be continuous.

The condition-1 demands that the exponentials are the same at the boundaries for all points and at all times for the three waves. That is possible, only when

$$\omega_2 = \omega_R = \omega_T = \omega \quad \rightarrow (4)$$

$$\text{and } \vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r} \quad \rightarrow (5).$$

Since, $k_I = k_R$, by eqn (5) $\rightarrow \boxed{\theta_I = \theta_R}$ $\theta_I \rightarrow \text{angle of incidence}$ $\theta_R \rightarrow \text{angle of reflection}$

and using $v = \frac{\omega}{k} = \sqrt{\epsilon \mu}$, we have got the Snell's law -

$$\left[\frac{\sin \theta_I}{\sin \theta_T} = \frac{n_2}{n_1} \right], \quad n_1, n_2 \rightarrow \text{r.i. of media} \\ 1 \& 2.$$

Where θ_T - angle of refraction $\rightarrow (6)$

The condition-2 on D_n and B_n are automatically satisfied, provided -

$$(\vec{E}_I + \vec{E}_R) \times \hat{e}_n = \vec{E}_T \times \hat{e}_n \quad \rightarrow (7)$$

$$\text{and } (\vec{H}_I + \vec{H}_R) \times \hat{e}_n = \vec{H}_T \times \hat{e}_n \quad \rightarrow (8)$$

$\hat{e}_n \rightarrow$ unit vector along the normal.

Applying eqn's ①, ② & ③ in ⑧ \rightarrow

$$⑧ \left(\frac{\vec{k}_I \times \vec{E}_I}{\mu, \omega_I} + \frac{\vec{k}_R \times \vec{E}_R}{\mu, \omega_R} \right) \times \hat{e}_n = \left(\frac{\vec{k}_T \times \vec{E}_T}{\omega_2 M_2} \right) \times \hat{e}_n$$

for non-magnetic material $\underline{\mu_1 \approx \mu_2}$ at all optical frequencies., the above eqn becomes -

$$(\vec{k}_I \times \vec{E}_I + \vec{k}_R \times \vec{E}_R) \times \hat{e}_n = (\vec{k}_T \times \vec{E}_T) \times \hat{e}_n \rightarrow ⑨$$

$$(\because \omega_I = \omega_R = \omega_T = \omega)$$

It is convenient to apply the boundary condition - 2 through the eqn's ⑦ & ⑨ considering two cases separately -

- ① the one pertaining the case when \vec{E}_I vector of the incident wave ~~vector~~ is perpendicular to (polarised to) the plane of incidence, and
- ② the other when \vec{E}_I vector is parallel to the ~~vector~~ plane of incidence.

Any general arbitrary case of the electric vector \vec{E}_I can be considered to arise from the linear combination of these two cases.

Let us take up the two cases one by one.

(9)

Case-I - \vec{E} is \perp^r to the plane of incidence (YZ)

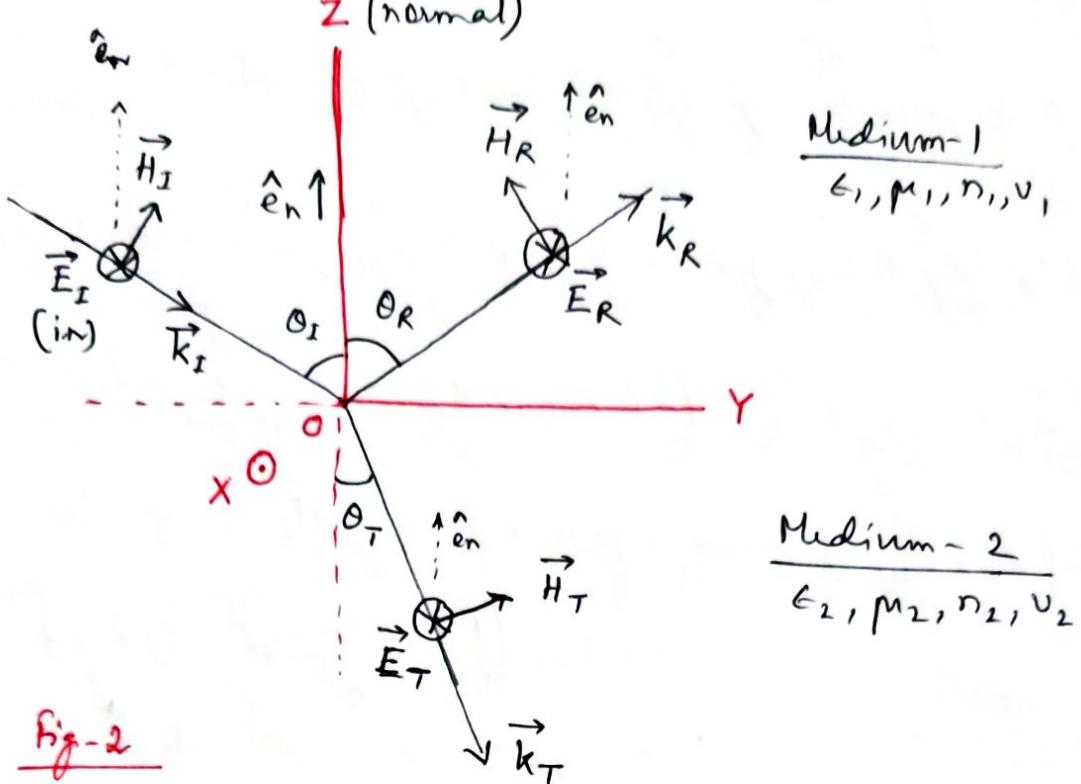
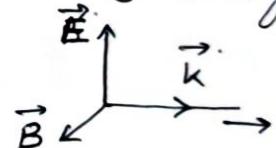


Fig-2

\vec{E} , \vec{B} and \vec{k} constitute a right-handed orthogonal set, if you rotate the screw from \vec{E} to \vec{B} through the smaller angle, the screw advances along the dirⁿ of propagation of the wave, i.e. along \vec{k} .



The electric field vectors \vec{E} are supposed to be directed ~~outward from~~ into the plane of incidence YZ corresponding to this case-I.

Now you apply the condition-2 through the two eqns in (7) & (9). Now using (1), (2) & (3) -

$$(7) \Rightarrow \vec{E}_{OI} \times \hat{e}_n + \vec{E}_{OR} \times \hat{e}_n = \vec{E}_{OT} \times \hat{e}_n \quad (\because \text{exponentials are equal by condn. (1)})$$

$$\Rightarrow E_{OI} \times 1 \times \sin\left(\frac{\pi}{2}\right) \hat{j} + E_{OR} \times 1 \times \sin\frac{\pi}{2} \hat{j} = E_{OT} \times 1 \times \sin\frac{\pi}{2} \hat{j}$$

\parallel \vec{E} 's are pointing along $-\hat{i}$ dirⁿ, \hat{e}_n pointing along \hat{k} -dirⁿ
(* Apply screw rule as in Fig-2)

(10)

$$\therefore \vec{E}_{OI} + \vec{E}_{OR} = \vec{E}_{OT} \quad \rightarrow (10).$$

Amplitudes of the incident, reflected and transmitted waves.

$$(9) \Rightarrow (\vec{k}_I \times \vec{E}_{OI}) \times \hat{e}_n + (\vec{k}_R \times \vec{E}_{OR}) \times \hat{e}_n = (\vec{k}_T \times \vec{E}_{OT}) \times \hat{e}_n$$

$$\therefore (\vec{k}_I \times \vec{E}_{OI}) \times \hat{e}_n = (k_I \times E_{OI} \times \sin \frac{\pi}{2}) \underset{\uparrow}{\hat{H}_I} \times \hat{e}_n$$

$\because (\vec{k}_I \times \vec{E}_{OI})$ vector is pointing along \vec{H}_I by eqn $\sim (16)$, apply screw rule.

$$= k_I E_{OI} [1 \times 1 \times \sin(\frac{\pi}{2} - \theta_I)] \hat{i}$$

$$= k_I E_{OI} \cos \theta_I \hat{i}$$

$(\hat{H}_I \times \hat{e}_n)$ vector is pointing along \hat{i} , apply screw rule.

$$\text{Similarly } (\vec{k}_R \times \vec{E}_{OR}) \times \hat{e}_n = k_R E_{OR} \cos \theta_R (-\hat{i})$$

$$\text{and } (\vec{k}_T \times \vec{E}_{OT}) \times \hat{e}_n = k_T E_{OT} \cos \theta_T \hat{i}$$

Hence, the above eqn becomes -

$$k_I E_{OI} \cos \theta_I - k_R E_{OR} \cos \theta_R = k_T E_{OT} \cos \theta_T$$

$$\Rightarrow k_I E_{OI} \cos \theta_I - k_I E_{OR} \cos \theta_I = k_T E_{OT} \cos \theta_T$$

(Using the outcome of boundary condition - 1)
 $\theta_I = \theta_R$

$$\text{or } (E_{OI} - E_{OR}) = \frac{k_T E_{OT} \cos \theta_T}{k_I} \quad \rightarrow (11)$$

$\cos \theta_I$

To find the ratio $\frac{E_{OR}}{E_{OI}}$ and $\frac{E_{OT}}{E_{OI}}$ required for the calculation of reflection and transmission

(11)

coefficients for non-conducting medium, we have to follow the mathematics given below. It will ^{also} lead us to formulate Fresnel's eqns for case-I.

Now,

$$(10) \times \frac{k_I \cos \theta_T}{k_L} = (11).$$

$$\Rightarrow (E_{OL} + E_{OR}) \times \frac{k_T \cos \theta_T}{k_I} = (E_{OL} - E_{OR}) \cos \theta_I$$

$$\Rightarrow E_{OL} \left[\frac{k_T}{k_I} \cos \theta_T - \cos \theta_I \right] = -E_{OR} \left[\cos \theta_L + \frac{k_I}{k_L} \cos \theta_T \right]$$

$$\Rightarrow \frac{E_{OR}}{E_{OL}} = \frac{\cos \theta_I - \frac{k_T}{k_I} \cos \theta_T}{\cos \theta_L + \frac{k_I}{k_L} \cos \theta_T}$$

$$\text{or } \frac{E_{OR}}{E_{OL}} = \frac{\cos \theta_I - \frac{n_2}{n_1} \cos \theta_T}{\cos \theta_L + \frac{n_2}{n_1} \cos \theta_T}$$

→ (12)a

$$v = \frac{\omega}{k}$$

$$\frac{k_T}{k_I} = \frac{\omega_T/v_2}{\omega_I/v_1}$$

$$= \frac{v_1}{v_2} \quad (\omega_I = \omega_T)$$

$$\therefore \boxed{\frac{k_T}{k_I} = \frac{n_2}{n_1}}$$

By applying Snell's law -

$$\frac{n_2}{n_1} = \frac{\sin \theta_I}{\sin \theta_T}$$

in eqnⁿ (12) →

$$\frac{E_{OR}}{E_{OL}} = \frac{\cos \theta_L - \frac{\sin \theta_I}{\sin \theta_T} \cos \theta_T}{\cos \theta_L + \frac{\sin \theta_I}{\sin \theta_T} \cos \theta_T}$$

$$= \frac{\sin \theta_T \cos \theta_I - \sin \theta_I \cos \theta_T}{\sin \theta_T \cos \theta_L + \sin \theta_I \cos \theta_T}$$

$$\text{or } \boxed{\frac{E_{OR}}{E_{OL}} = \frac{\sin(\theta_T - \theta_I)}{\sin(\theta_T + \theta_I)}} \rightarrow (12b)$$

(12)

$$(10) \times \cos \theta_I + (11) \Rightarrow$$

$$2 E_{OI} \cos \theta_I = E_{OT} \left[\cos \theta_I + \frac{k_T}{k_I} \cos \theta_T \right]$$

$$\Rightarrow \frac{E_{OT}}{E_{OI}} = \frac{2 \cos \theta_I}{\cos \theta_I + \frac{n_2}{n_1} \cos \theta_T} \rightarrow (13a)$$

Applying Snell's law in (13a) -

$$\frac{\cancel{E_{OT}}}{\cancel{E_{OI}}} \frac{E_{OT}}{E_{OI}} = \frac{2 \cos \theta_I}{\cos \theta_I + \frac{\sin \theta_I}{\sin \theta_T} \cos \theta_T}$$

$$\therefore \boxed{\frac{E_{OT}}{E_{OI}} = \frac{2 \sin \theta_T \cos \theta_I}{\sin(\theta_T + \theta_I)}} \rightarrow (13b)$$

Eqs (12b) and (13b) are called Fresnel's eqns for oblique incidence, when \vec{E} is \perp^r upon the plane of incidence, giving the ratios of the amplitudes of the reflected and transmitted waves to the incident wave.

* For normal incidence $\theta_I = \theta_R = \theta_T = 0^\circ$

$$(12b) \rightarrow \frac{E_{OR}}{E_{OI}} = \frac{n_1 - n_2}{n_1 + n_2} \rightarrow (12c)$$

$$(13a) \rightarrow \frac{E_{OT}}{E_{OI}} = \frac{2n_1}{n_1 + n_2} \rightarrow (13d)$$

which are again Fresnel's eqns for normal incidence, when \vec{E} is \perp^r to the plane of incidence.

* If $n_2 > n_1$, by Snell's law $\theta_T < \theta_I$ and (13)

$$\left[\frac{n_2}{n_1} = \frac{\sin \theta_I}{\sin \theta_T}, \text{ greater the value of } \theta, \text{ larger is the sine value.} \right]$$

by Fresnel's eqn (12b) or (12c), $\frac{E_{0R}}{E_{0I}}$ must be -ve ($\because \sin(-\theta) = -\sin \theta$).

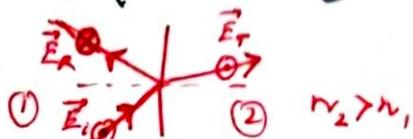
It indicates that the \vec{E}_R (the electric field vector of reflected wave) oscillates 180° out of phase with \vec{E}_I (electric field vector of incident wave)

~~at the boundary of diff媒質~~ when the e-m wave incident from rarer to a at the boundary surface of a denser medium.

* If $n_2 < n_1 \rightarrow \theta_T > \theta_I$ and by Fresnel's eqn (12b) or (12c), $\frac{E_{0R}}{E_{0I}}$ is +ve, indicating that there will not occur any change of phase b/w \vec{E}_I and \vec{E}_R .

* The ratio $\frac{E_{0T}}{E_{0I}}$ is always +ve irrespective of $n_2 > n_1$ or $n_2 < n_1$ (by Fresnel's eqn (13b) or (13c)) indicating that there is no phase change.

b/w \vec{E}_I (electric field vector of incident wave) and \vec{E}_T (electric field vector of transmitted wave)



→ →

Reflection Coefficient (R) -

The reflected coefficient R is defined as the energy flux get reflected from boundary surface normally in unit time divided by the energy flux incident on that boundary surface normally in unit time. Mathematically -

$$R_{\perp} = \frac{|\hat{e}_n \cdot \vec{S}_R|}{|\hat{e}_n \cdot \vec{S}_I|}$$

$$= \left| \frac{\hat{e}_n \cdot \hat{k}_R \left| \vec{E}_R \times \vec{H}_R \right|}{\hat{e}_n \cdot \hat{k}_I \left| \vec{E}_I \times \vec{H}_I \right|} \right|$$

$$= \left| \frac{\cos \theta_R}{-\cos \theta_I} \frac{\left| \vec{E}_R \times (\vec{k}_R \times \vec{E}_R) / w_R \mu_1 \right|}{\left| \vec{E}_I \times (\vec{k}_I \times \vec{E}_I) / w_I \mu_2 \right|} \right|$$

$$= \left| \frac{\vec{k}_R (\vec{E}_R \cdot \vec{E}_R) - \vec{E}_R (\vec{E}_R \cdot \vec{k}_R)}{\vec{k}_I (\vec{E}_I \cdot \vec{E}_I) - \vec{E}_I (\vec{E}_I \cdot \vec{k}_I)} \right|$$

$$= \left| \frac{k_R E_R^2}{k_I E_I^2} \right|$$

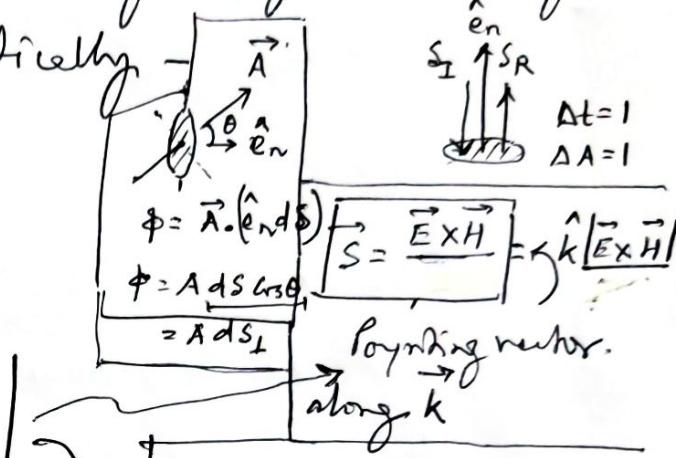
$$R_{\perp} = \left| \frac{E_{0R}}{E_{0I}} \right|^2$$

($\because k_I = k_R$)

14Q.

: (in same medium - 1) $|A|^2 = A^* A$

[The subscript \perp in R indicates that \vec{E} is \perp^{r} upon the plane of incidence.]



From fig-2

$$\hat{e}_n \cdot \hat{k}_R = 1 \times 1 \times \cos \theta_R$$

$$\hat{e}_n \cdot \hat{k}_I = 1 \times 1 \times \cos(\pi - \theta_I)$$

$$= -\cos \theta_I$$

by eqn's ① & ②

Applying - [BAC - CAB] Rule
for vector triple product
and $\theta_I = \theta_R$, $w_I = w_R$

$\vec{E}, \vec{B}, \vec{k} \rightarrow$ orthogonal
right handed system formed
 $\therefore \vec{E} \cdot \vec{k} = 0$.

(16)

Using Fresnel's eqn (12b), the reflection coefficient for oblique incidence \rightarrow

$$R_{\perp} = \frac{\sin^2(\theta_T - \theta_I)}{\sin^2(\theta_T + \theta_I)}$$

$$\rightarrow 14b$$

And for normal incidence, by using (12c) in (14a) \rightarrow

$$R_{\perp} = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

$$\rightarrow 14c$$

Transmission Coefficient -

The transmission coefficient (T) is defined as the energy flux transmitted into the second medium through unit area of the boundary surface in unit time divided by the energy flux that incident on unit area of the boundary in unit time, namely. Mathematically

$$T_{\perp} = \left| \frac{\hat{e}_n \cdot \vec{s}_T}{\hat{e}_n \cdot \vec{s}_I} \right|$$

Magnitude required

$$= \left| \frac{\hat{e}_n \cdot \hat{k}_T (\vec{E}_T \times \vec{H}_T)}{\hat{e}_n \cdot \hat{k}_I (\vec{E}_I \times \vec{H}_I)} \right|$$

$\text{fig 2} \rightarrow$
 $\hat{e}_n \cdot \hat{k}_T = 1 \times 1 \times \cos(\pi - \theta_T)$
 $= \cos \theta_T$

$$= \left| \frac{-\cos \theta_T}{-\cos \theta_I} \cdot \frac{|\vec{E}_T \times (\vec{k}_T \times \vec{E}_T)| / \mu_2 \omega_T}{|\vec{E}_I \times (\vec{k}_I \times \vec{E}_I)| / \mu_1 \omega_I} \right|$$

$$= \left| \frac{\cos \theta_T}{\cos \theta_I} \cdot \frac{k_T E_{0T}^2}{k_I E_{0I}^2} \right|$$

$(\mu_1 \approx \mu_2, \omega_I = \omega_T)$
Applies same trick as before

(17)

$$T_{\perp} = \frac{\cos \theta_T}{\cos \theta_I} \frac{n_2}{n_1} \left| \frac{E_{0T}^2}{E_{0I}^2} \right|$$

if $0^\circ \leq \theta \leq 90^\circ$
 $1 \leq \cos \theta \leq 0$
 \downarrow
+ve

For oblique incidence,

$$\therefore T_{\perp} = \frac{\cos \theta_T}{\cos \theta_I} \frac{\sin \theta_I}{\sin \theta_T} \times \left| \frac{2 \sin \theta_T \cos \theta_I}{\sin(\theta_T + \theta_I)} \right|^2$$

(using Snell's law and Fresnel's eqn (13b))
for oblique incidence when \vec{E} electric field vector in 1st to plane of incidence

$$= \frac{(\cos \theta_T \sin \theta_I)(4 \sin \theta_T \cos \theta_I)}{\sin^2(\theta_T + \theta_I)}$$

$$T_{\perp} = \frac{\sin 2\theta_T \sin 2\theta_I}{\sin^2(\theta_T + \theta_I)} \rightarrow (15b)$$

For normal incidence, using (13c) in (15a) →

$$T_{\perp} = \frac{\cancel{\cos \theta_T}}{\cancel{\cos \theta_I}} \frac{n_2}{n_1} \left(\frac{2n_1}{n_1 + n_2} \right)^2 \quad | \quad \theta_I = \theta_T = 0^\circ$$

→ (15c)

Using eqns (14b) and (15b), it can be shown that

$$[R_{\perp} + T_{\perp} = 1] \quad (HW) \rightarrow \text{for oblique incidence}$$

→ for normal incidence

(Logically - Energy in incident wave = Energy in reflected wave + Energy in transmitted wave)



(18)

Case-II - \vec{E} is parallel to the plane of incidence (yz)

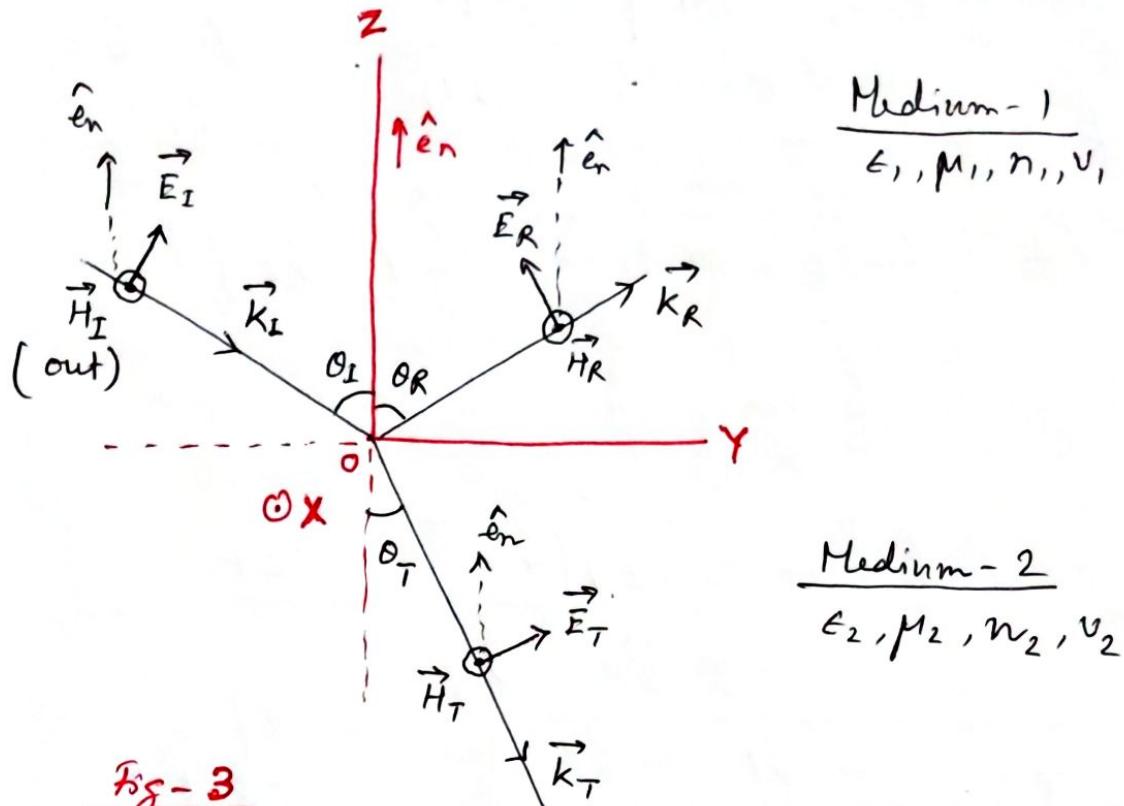


Fig-3

The electric field vectors \vec{E} are supposed to be parallel to the plane of incidence (yz). Now we will apply boundary condition-2 through the eqns (7) & (8) in this present case.

$$(7) \Rightarrow \vec{E}_{0L} \times \hat{e}_n + \vec{E}_{0R} \times \hat{e}_n = \vec{E}_{0T} \times \hat{e}_n$$

(\because amplitudes in (1), (2) & (3) for incident, reflected and transmitted waves are same by the boundary condition-1)

$$\begin{aligned} \therefore E_{0L} \times 1 \times \sin\left(\frac{\pi}{2} - \theta_L\right) \hat{i} + E_{0R} \times 1 \times \sin\left(\frac{\pi}{2} - \theta_R\right) (-\hat{i}) \\ = E_{0T} \times 1 \times \sin\left(\frac{\pi}{2} - \theta_T\right) \hat{i} \end{aligned}$$

(\vec{E} vectors are lying on yz plane, \hat{e}_n is pointing along \hat{k} dir as in fig-3. Apply screw rule also.)

$$\therefore \boxed{(E_{0I} - E_{0R}) \cos \theta_I = E_{0T} \cos \theta_T} \quad (\because \theta_I = \theta_R) \quad (17)$$

$$⑨ \Rightarrow (\vec{k}_I \times \vec{E}_{0I}) \times \hat{e}_n + (\vec{k}_R \times \vec{E}_{0R}) \times \hat{e}_n = (\vec{k}_T \times \vec{E}_{0T}) \times \hat{e}_n$$

$$\Rightarrow \underbrace{(k_I E_{0I} \sin \frac{\pi}{2} \hat{i}) \times \hat{e}_n}_{+} + (k_R E_{0R} \sin \frac{\pi}{2} \hat{i}) \times \hat{e}_n = (k_T E_{0T} \sin \frac{\pi}{2} \hat{i}) \times \hat{e}_n$$

$$\Rightarrow k_I (E_{0I} + E_{0R}) = k_T E_{0T} \quad (\because k_I = k_R)$$

$$\Rightarrow \boxed{E_{0I} + E_{0R} = \frac{k_T}{k_I} E_{0T}} \quad (18)$$

Using eqns ⑦ & ⑧, we will find out the two ratios $\frac{E_{0R}}{E_{0I}}$ & $\frac{E_{0T}}{E_{0I}}$ to formulate Fresnel's eqns. and coefficients of reflection and transmission for oblique incidence, when \vec{k} is ~~perpendicular~~ ~~perp~~ to plane of incidence.

$$(17) \times \frac{k_I}{k_I} = (18) \times \cos \theta_T$$

$$\Rightarrow (E_{0I} - E_{0R}) \frac{k_T}{k_I} = (E_{0I} + E_{0R}) \cos \theta_T$$

$$\Rightarrow E_{0I} \left(\frac{k_T}{k_I} \cos \theta_I - \cos \theta_T \right) = E_{0R} \left(\frac{k_T}{k_I} \cos \theta_I + \cos \theta_T \right)$$

$$\therefore \frac{E_{0R}}{E_{0I}} = \frac{\frac{k_T}{k_I} \cos \theta_I - \cos \theta_T}{\frac{k_T}{k_I} \cos \theta_I + \cos \theta_T}$$

(20)

$$\left| \frac{E_{OR}}{E_{OI}} = \frac{\frac{n_2}{n_1} \cos \theta_I - \cos \theta_T}{\frac{n_2}{n_1} \cos \theta_I + \cos \theta_T} \right|$$

19a

$$\begin{aligned} \text{or } \frac{E_{OR}}{E_{OI}} &= \frac{\frac{\sin \theta_I}{\sin \theta_T} \cos \theta_I - \cos \theta_T}{\frac{\sin \theta_I}{\sin \theta_T} \cos \theta_I + \cos \theta_T} \\ &= \frac{\sin 2\theta_I - \sin 2\theta_T}{\sin 2\theta_I + \sin 2\theta_T} \\ &= \frac{2 \cos(\theta_I + \theta_T) \sin(\theta_I - \theta_T)}{2 \sin(\theta_I + \theta_T) \cos(\theta_I - \theta_T)} \\ \therefore \left| \frac{E_{OR}}{E_{OT}} = \frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)} \right| \end{aligned}$$

Applying Snell's law

$$\sin A = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A + \sin B$$

$$\begin{aligned} &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \\ &= 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \end{aligned}$$

19b

$$\text{Now } (17) + (18) \times \cos \theta_2 \Rightarrow$$

$$2 E_{OI} \cos \theta_I = E_{OT} \left(\cos \theta_T + \frac{k_T}{k_I} \cos \theta_2 \right)$$

$$\rightarrow \left| \frac{E_{OT}}{E_{OI}} = \frac{2 \cos \theta_I}{\cos \theta_T + \frac{n_2}{n_1} \cos \theta_2} \right|$$

20a

Applying Snell's law -

$$\frac{E_{UT}}{E_{OI}} = \frac{2 \cos \theta_I}{\cos \theta_T + \frac{\sin \theta_I}{\sin \theta_T} \cos \theta_2}$$

$$\therefore \frac{E_{OT}}{E_{OI}} = \frac{4 \sin \theta_T \cos \theta_I}{\sin 2\theta_T + \sin 2\theta_I}$$

$$\left| \frac{E_{OT}}{E_{OI}} = \frac{2 \sin \theta_T \cos \theta_I}{\sin(\theta_I + \theta_T) \cos(\theta_I - \theta_T)} \right|$$

20b

(21)

Equations (19b) and (20b) are called the
Fresnel's equations for oblique incidence, when
 \vec{E}_I is parallel to the plane of incidence,
giving the ratios of the amplitude of reflected
wave to incident wave and that of transmitted
wave to incident wave.

Fresnel's eqn's for normal incidence ($\theta_2 = \theta_R = \theta_T = 0^\circ$)

$$(19a) \rightarrow \boxed{\frac{E_{0R}}{E_{0I}} = \frac{n_2 - n_1}{n_2 + n_1}}$$

(19c)

$$(20a) \rightarrow \boxed{\frac{E_{0T}}{E_{0I}} = \frac{2n_1}{n_1 + n_2}}$$

(20c)

(where \vec{E} is parallel to the plane of incidence)

(22)

Reflection Coefficient -

(Defⁿ already given in page-15)

$$R_{11} = \left| \frac{\hat{e}_n \cdot \vec{s}_R}{\hat{e}_n \cdot \vec{s}_I} \right|$$

$$= \left| \frac{\hat{e}_n \cdot \hat{k}_R (\vec{E}_R \times \vec{H}_R)}{\hat{e}_n \cdot \hat{k}_I (\vec{E}_I \times \vec{H}_I)} \right|$$

$$= \left| \frac{\cos \theta_R}{-\cos \theta_I} \frac{|\vec{E}_R \times (\vec{k}_R \times \vec{E}_R)| / w_R \mu_r}{|\vec{E}_I \times (\vec{k}_I \times \vec{H}_R)| / w_I \mu_i} \right|$$

$$= \left| \frac{\cos \theta_R}{\cos \theta_I} \frac{|\vec{k}_R (\vec{E}_R \cdot \vec{E}_R) - \vec{E}_R (\vec{E}_R \cdot \vec{k}_R)|}{|\vec{k}_I (\vec{E}_I \cdot \vec{E}_R) - \vec{E}_I (\vec{E}_R \cdot \vec{k}_I)|} \right|$$

$$= \left| \frac{E_R^2}{E_I^2} \right|$$

$$R_{11} = \left| \frac{E_{0R}}{E_{0I}} \right|^2 \rightarrow (21a)$$

$$\vec{s} = \frac{\vec{E} \times \vec{H}}{\omega M} = \hat{k} |\vec{E} \times \vec{H}|$$

From $\vec{J} \cdot \vec{s} = 0$

$$\hat{e}_n \cdot \hat{k}_R = 1 \times 1 \times \cos \theta_R$$

$$\hat{e}_n \cdot \hat{k}_I = 1 \times 1 \times \cos(\pi - \theta_I) \\ = -\cos \theta_I$$

Apply

$$\vec{A} \times (\vec{B} \times \vec{C})$$

$$= \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})$$

$\rightarrow [BAC - CAB]$ Rule

$$\& w_I = w_R, \theta_I = \theta_R,$$

$\vec{E}, \vec{B}, \vec{k} \rightarrow$ right handed
orthogonal system

$$\text{so, } \vec{E} \cdot \vec{k} = 0$$

$k_I = k_R$ in medium-1

Applying (19b) in the above eqn (fresnel's eqn for oblique incidence), where \vec{E} is parallel to the plane of incidence)

$$R_{11} = \left| \frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)} \right|^2 \rightarrow (21b)$$

And for normal incidence by using eqn (19c)

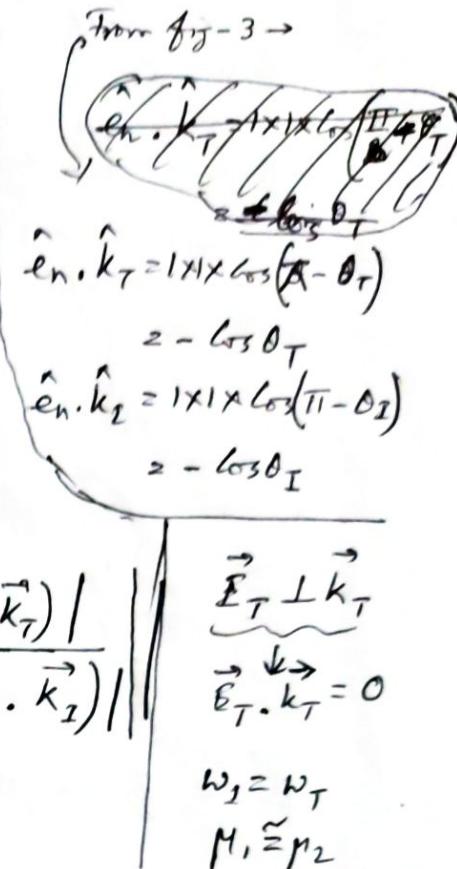
$$R_{11} = \left(\frac{n_2 - n_1}{n_2 + n_1} \right)^2 \rightarrow (21c)$$

(23)

Transmission Coefficient -

Defn - already given in page - (10).

$$\begin{aligned}
 T_{II} &= \left| \frac{\hat{n} \cdot \vec{s}_T}{\hat{n} \cdot \vec{s}_I} \right| \\
 &= \left| \frac{\hat{n} \cdot \hat{k}_T (\vec{E}_T \times \vec{H}_T)}{\hat{n} \cdot \hat{k}_I (\vec{E}_I \times \vec{H}_I)} \right| \\
 &= \left| \frac{-\cos \theta_T}{-\cos \theta_I} \frac{\vec{E}_T \times (\vec{k}_T \times \vec{E}_T) / \mu_T \mu_2}{\vec{E}_I \times (\vec{k}_I \times \vec{E}_I) / \mu_I \mu_1} \right| \\
 &= \frac{\cos \theta_T}{\cos \theta_I} \left| \frac{\vec{k}_T (\vec{E}_T \cdot \vec{E}_T) - \vec{E}_T (\vec{E}_T \cdot \vec{k}_T)}{\vec{k}_I (\vec{E}_I \cdot \vec{E}_I) - \vec{E}_I (\vec{E}_I \cdot \vec{k}_I)} \right| \\
 &= \frac{\cos \theta_T}{\cos \theta_I} \left| \frac{k_T E_T^2}{k_I E_I^2} \right|
 \end{aligned}$$



$$\boxed{T_{II} = \frac{\cos \theta_T}{\cos \theta_I} \frac{n_2}{n_1} \left| \frac{E_{0T}}{E_{0I}} \right|^2} \rightarrow (22a)$$

(20b)

Applying Snell's law and Fresnel's eqns for
Obligatory incidence, when \vec{E} (electric field vector) is parallel
to the plane of incidence -

$$T_{II} = \frac{\cos \theta_T}{\cos \theta_I} \frac{\sin \theta_I}{\sin \theta_T} \frac{4 \sin^2 \theta_T \cos^2 \theta_I}{\sin(\theta_I + \theta_T) \cos(\theta_I - \theta_T)}$$

$$\boxed{T_{II} = \frac{\sin 2\theta_I \sin 2\theta_T}{\sin(\theta_I + \theta_T) \cos(\theta_I - \theta_T)}} \rightarrow (22b)$$

(24)

And for normal incidence applying (20c) in (22a)

$$T_{II} = \frac{n_2}{n_1} \left(\frac{2n_1}{n_1 + n_2} \right)^2 \quad \left(\theta_I = \theta_r = 0^\circ \right)$$

(22c)

when \vec{E} is parallel to the plane of incidence.

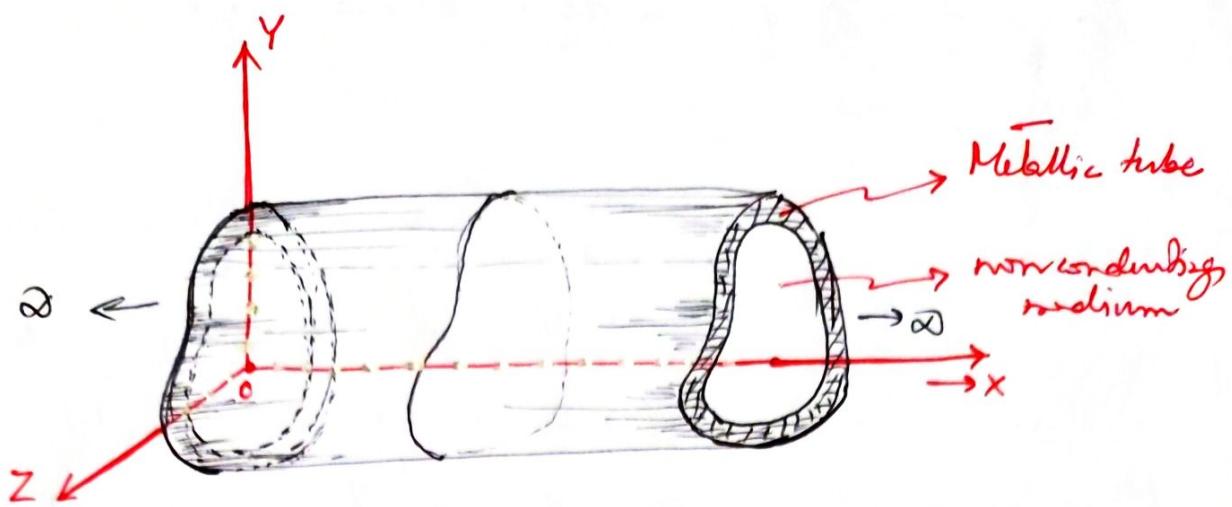
Propagation of e-m waves in bounded media,

Wave Guide.

- Basic concepts of waveguides -

Let us consider a region of nonconducting medium (say, vacuum or air) bounded by an infinitely long tube made of conducting material (say metal) and of arbitrary cross-sectional shape (the cross-sectional area and shape of the tube being uniform throughout). The non-conducting bounded region is assumed to be a loss-less region for e-m waves. Again, since the wall of the conducting tube is of perfect conductor ($\sigma \rightarrow \infty$), the skin depth ($\delta = \sqrt{\frac{2}{\omega \mu \sigma}}$) is very very small. So, a thin coating of a good conductor (like silver) serves the basic purpose of the metallic tube. Such a structure, i.e. a hollow cylindrical tube filled with a non-conducting medium can guide the transmission of e-m energy along the axis of the tube and such structure is called a wave guide. Thus a wave guide is a cylindrical pipe/ bounded by a conductor of high conductivity.

(2)

Fig-1

Transmission of e-m waves using wave guides over short distances is the most efficient way of transmission. Wave guides were evolved and used in practical communication electronics and now, wave guides has assumed great importance in optoelectronics.

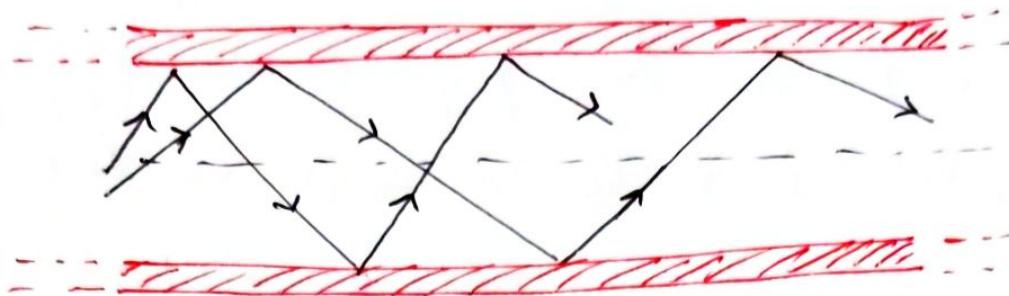
The two most common wave guides are those with ① rectangular cross-section and ② circular cross-sections. But we start our discussion with a wave guide of arbitrary cross-sectional shape (fig-1).

To study the propagation of e-m waves down a wave guide, one can adopt any one approach out of the two listed below.

Approach-1 - In this approach, the propagation of e-m waves is assumed to be the result of successive reflections from the metallic boundaries.

(3)

and the resultant of various reflected modes is calculated through the principle of superposition of their amplitudes.



Approach - 2 — In this approach, general solutions of the wave-equations for the e.f. \vec{E} and m.f. \vec{B} inside the waveguide are sought with the application of Maxwell's eqns and appropriate boundary conditions on the field vectors \vec{E} and \vec{B} .

The second approach will be followed in our discussion in view of its simplicity and instructiveness.

(4)

EM waves (guided waves) propagating along a wave guide -

We consider a general form of wave guide - a hollow conducting pipe of arbitrary cross-section (fig-1). The interior is assumed to be ~~discontinuous~~ ^{discontinuous} and the bounding walls are assumed to be perfect conductor ($\sigma \rightarrow \infty$) and the shape and area of the cross-section ^{are} uniform along its axis.

Maxwell's eqn's in the interior of the wave guide -

$$\vec{D} \cdot \vec{E} = 0 \rightarrow \textcircled{1} \quad \vec{D} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \textcircled{3} \quad f=0.$$

$$\vec{D} \cdot \vec{B} = 0 \rightarrow \textcircled{2} \quad \vec{D} \times \vec{B} = \mu c \frac{\partial \vec{E}}{\partial t} \rightarrow \textcircled{4} \quad \vec{J}=0$$

We are interested in monochromatic e.m. wave that propagates down the tube along ^{its axis, i.e.} $+x$ -direction. (fig-1), so \vec{E} and \vec{B} have the general form

$$\vec{E}(x, y, z, t) = \vec{E}_0^{(y, z)} e^{j(k_x x - \omega t)} \rightarrow \textcircled{5}$$

$$\vec{B}(x, y, z, t) = \vec{B}_0^{(y, z)} e^{j(k_x x - \omega t)} \rightarrow \textcircled{6}$$

The e.f. and m.f. in $\textcircled{5}$ & $\textcircled{6}$ must satisfy Maxwell's eqn's. Again, the \vec{E}_0 and \vec{B}_0 are separated into components parallel to and transverse to the axis (x -axis) of the wave guide.

$$\vec{E}_0(y, z) = \underbrace{i E_{0x}(y, z)}_{\vec{E}_{0L}} + \underbrace{j E_{0y}(y, z)}_{\vec{E}_{0T}} + \underbrace{k E_{0z}(y, z)}_{\vec{E}_{0T}} \quad \text{--- (7)} \quad (5)$$

$$\text{and } \vec{B}_0(y, z) = \underbrace{i B_{0x}(y, z)}_{\vec{B}_{0L}} + \underbrace{j B_{0y}(y, z)}_{\vec{B}_{0T}} + \underbrace{k B_{0z}(y, z)}_{\vec{B}_{0T}}. \quad \text{--- (8)}$$

Putting (5) & (6) in (3) and (4), the new set of Maxwell's eqn's will be -

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \text{--- (1')} \quad \vec{\nabla} \times \vec{E} = j\omega \vec{B} \rightarrow (3')$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{--- (2')} \quad \vec{\nabla} \times \vec{B} = -j\mu_0 \epsilon \omega \vec{E} \rightarrow (4')$$

Now putting (7) & (8) in (3')

$$i \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - j \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) + k \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = j\omega \left(i B_{0x} + j B_{0y} + k B_{0z} \right)$$

X-component

$$\therefore \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = j\omega B_{0x}$$

$$\Rightarrow \frac{\partial}{\partial y} \left(B_{0z} e^{j(k_x x - \omega t)} \right) - \frac{\partial}{\partial z} \left(B_{0y} e^{j(k_x x - \omega t)} \right) = j\omega B_{0x} e^{j(k_x x - \omega t)}$$

$$\Rightarrow \boxed{\frac{\partial B_{0z}}{\partial y} - \frac{\partial B_{0y}}{\partial z} = j\omega B_{0x}} \rightarrow (9a)$$

(6)

$$\frac{\partial \text{- component}}{\partial z} - \left(\underbrace{\frac{\partial}{\partial x} \left(E_{0z} e^{j(k_x x - \omega t)} \right)}_{\rightarrow} \right) - \frac{\partial}{\partial z} \left(E_{0x} e^{j(k_x x - \omega t)} \right) = j \omega B_{0y} e^{j(k_x x - \omega t)}$$

$$\Rightarrow -E_{0z} (jk_x) + \frac{\partial E_{0x}}{\partial z} = j \omega B_{0y}$$

$$\Rightarrow \boxed{\frac{\partial E_{0x}}{\partial z} - j k_x E_{0z} = j \omega B_{0y}} \rightarrow (9b)$$

$$\frac{2\text{- component } 3}{\frac{\partial}{\partial x} \left(B_{0y} e^{j(k_x x - \omega t)} \right)} - \frac{\partial}{\partial y} \left(E_{0x} e^{j(k_x x - \omega t)} \right) = j \omega B_{0z} e^{j(k_x x - \omega t)}$$

$$\Rightarrow B_{0y} (jk_x) - \frac{\partial E_{0x}}{\partial y} = j \omega B_{0z}$$

$$\Rightarrow \boxed{j k_x B_{0y} - \frac{\partial E_{0x}}{\partial y} = j \omega B_{0z}} \rightarrow (9c)$$

Now putting (7) & (8) in (4') \rightarrow

(Since (4') is symmetric with (3'), so changing $\vec{E} \leftrightarrow \vec{B}$
and $\omega \rightarrow \mu \epsilon \omega$ in (3'), (4') can be obtained)

$$\boxed{\frac{\partial B_{0z}}{\partial y} - \frac{\partial B_{0y}}{\partial z} = -j \mu \epsilon \omega E_{0x}} \rightarrow (10a)$$

$$\boxed{\frac{\partial B_{0x}}{\partial z} - j k_x B_{0z} = -j \mu \epsilon \omega E_{0y}} \rightarrow (10b)$$

$$\boxed{j k_x B_{0y} - \frac{\partial B_{0x}}{\partial y} = -j \mu \epsilon \omega E_{0z}} \rightarrow (10c)$$

With these eqns of (E_{0x}, E_{0y}, E_{0z}) and (B_{0x}, B_{0y}, B_{0z}) , our aim is to find out $E_{0y}, E_{0z}, B_{0x} \& B_{0z}$, in terms of $E_{0x} \& B_{0x}$ using the above eqns (9a - 9c) & (10a - 10c).

First we will find out $E_{0y} \& E_{0z}$ —

(9c) in (10b) \rightarrow

$$(10b) \rightarrow \frac{\partial B_{0x}}{\partial z} - k_x \frac{1}{\omega} \left(j k_x E_{0y} - \frac{\partial B_{0x}}{\partial y} \right) = -j \mu \epsilon \omega E_{0y}$$

$$= j \omega B_{0x} \rightarrow (9c)$$

$$\Rightarrow \frac{\partial B_{0x}}{\partial z} - j \frac{k_x^2}{\omega} E_{0y} + \frac{k_x}{\omega} \frac{\partial B_{0x}}{\partial y} = -j \mu \epsilon \omega E_{0y}$$

$$\Rightarrow \frac{k_x}{\omega} \frac{\partial B_{0x}}{\partial y} + \frac{\partial B_{0x}}{\partial z} = j \left(\mu \epsilon \omega + \frac{k_x^2}{\omega} \right) E_{0y}$$

$$\Rightarrow k_x \frac{\partial B_{0x}}{\partial y} + \omega \frac{\partial B_{0x}}{\partial z} = j \left(\frac{-\omega^2}{v^2} + k_x^2 \right) E_{0y} \quad \left| v = \frac{1}{\sqrt{\mu \epsilon}} \right.$$

$$\Rightarrow E_{0y} = \frac{j}{j \left(k_x^2 - \frac{\omega^2}{v^2} \right)} \left(k_x \frac{\partial B_{0x}}{\partial y} + \omega \frac{\partial B_{0x}}{\partial z} \right)$$

$$\therefore \boxed{E_{0y} = \left(\frac{j}{\frac{\omega^2}{v^2} - k_x^2} \right) \left(k_x \frac{\partial B_{0x}}{\partial y} + \omega \frac{\partial B_{0x}}{\partial z} \right)} \rightarrow (11a)$$

(9b) in (10c) \rightarrow

$$j k_x \frac{1}{j \omega} \left(\frac{\partial B_{0x}}{\partial z} - j k_x E_{0z} \right) - \frac{\partial B_{0x}}{\partial y} = -j \mu \epsilon \omega E_{0z}$$

$$\Rightarrow \frac{k_x}{\omega} \frac{\partial B_{0x}}{\partial z} - j \frac{k_x^2}{\omega} E_{0z} - \frac{\partial B_{0x}}{\partial y} = -j \mu \epsilon \omega E_{0z}$$

$$\Rightarrow \frac{k_x}{\omega} \frac{\partial B_{0x}}{\partial z} - \frac{\partial B_{0x}}{\partial y} = j \left(\frac{k_x^2}{\omega} - \mu \epsilon \omega \right) E_{0z}$$

$$\therefore \boxed{E_{0z} = \left(\frac{j}{\frac{\omega^2}{v^2} - k_x^2} \right) \left(k_x \frac{\partial B_{0x}}{\partial z} - \omega \frac{\partial B_{0x}}{\partial y} \right)} \rightarrow (11b)$$

Next, our task is to find out B_{0y} & B_{0z} .

(9)

(11b) in (9b) \rightarrow

$$\frac{\partial E_{0x}}{\partial z} - j k_x \left(\frac{j}{\frac{\omega}{v} - k_n^2} \right) \left(k_x \frac{\partial B_{0x}}{\partial z} - \omega \frac{\partial B_{0x}}{\partial y} \right) = j \omega B_{0y}$$

$$\Rightarrow \left\{ 1 + \left(\frac{k_n^2}{\frac{\omega}{v} - k_n^2} \right) \right\} \frac{\partial E_{0x}}{\partial z} - \left(\frac{k_x \omega}{\frac{\omega}{v} - k_n^2} \right) \frac{\partial B_{0x}}{\partial y} = j \omega B_{0y}$$

$$\Rightarrow \frac{\frac{\omega}{v}}{\frac{\omega}{v} - k_n^2} \frac{\partial E_{0x}}{\partial z} - \frac{k_x \omega}{\frac{\omega}{v} - k_n^2} \frac{\partial B_{0x}}{\partial y} = j \omega B_{0y}$$

$$\therefore \boxed{B_{0y} = \frac{j}{\left(\frac{\omega}{v} - k_n^2 \right)} \left(k_x \frac{\partial B_{0x}}{\partial y} - \frac{\omega}{v} \frac{\partial E_{0x}}{\partial z} \right)} \rightarrow (11c)$$

(11a) in (9c) \rightarrow

$$j k_x \left(\frac{j}{\frac{\omega}{v} - k_n^2} \right) \left(k_x \frac{\partial B_{0x}}{\partial y} + \omega \frac{\partial B_{0x}}{\partial z} \right) - \frac{\partial E_{0x}}{\partial z} = j \omega B_{0z}$$

$$\Rightarrow \frac{-\frac{\omega}{v}}{\frac{\omega}{v} - k_n^2} \frac{\partial E_{0x}}{\partial y} - \frac{k_x \omega}{\frac{\omega}{v} - k_n^2} \frac{\partial B_{0x}}{\partial z} = j \omega B_{0z}$$

$$\Rightarrow \boxed{B_{0z} = \frac{j}{\left(\frac{\omega}{v} - k_n^2 \right)} \left(k_x \frac{\partial B_{0x}}{\partial z} + \frac{\omega}{v} \frac{\partial E_{0x}}{\partial y} \right)} \rightarrow (11d)$$

If we know E_{0x} & B_{0x} longitudinal components of the e.f. & m.f. vectors of the guided wave propagating along x-dirn/axis of the cylindrical wave guide), the transverse components

$\vec{E}_T (E_y, E_z)$ and $\vec{B}_T (B_y, B_z)$ could be calculated quickly using the eqns in (11) so, now we have required transverse derivatives for E_{0x} & B_{0x} .

(10) Maxwell's eqns in (3') & (4') are applied to find out E_{ox} , E_{oy} and B_{ox} & B_{oy} (~~transverse~~ components of E_T and B_T) in terms of E_{ox} & B_{ox} (longitudinal components of e.f. and m.f. ratios of the wave). Now we are going to use the other two Maxwell's eqns (1') & (2'). To formulate some differential eqns for E_{ox} and B_{ox} .

Applying (11a) & (11b) in (1') -

$$(1') \rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0.$$

$$\Rightarrow \frac{\partial}{\partial x} \left\{ E_{ox}(y, z) e^{j(k_x x - \omega t)} \right\} + \frac{\partial}{\partial y} \left\{ E_{oy}(y, z) e^{j(k_x x - \omega t)} \right\} \\ + \frac{\partial}{\partial z} \left\{ E_{oz}(y, z) e^{j(k_x x - \omega t)} \right\} = 0 \\ \Rightarrow j k_x E_{ox} \cdot e^{j(k_x x - \omega t)} + \left(\frac{\partial E_{oy}}{\partial y} \right)_x e^{j(k_x x - \omega t)} \\ + \left(\frac{\partial E_{oz}}{\partial z} \right)_x e^{j(k_x x - \omega t)} = 0.$$

$$\Rightarrow j k_x E_{ox} + \frac{\partial E_{oy}}{\partial y} + \frac{\partial E_{oz}}{\partial z} = 0. \quad \rightarrow (A).$$

$$\Rightarrow j k_x E_{ox} + \left(\frac{j}{\frac{\omega}{v} - k_x^2} \right) \left\{ k_x \frac{\partial^2 E_{ox}}{\partial y^2} + \cancel{\omega \frac{\partial^2 B_{ox}}{\partial y \partial z}} + k_x \frac{\partial^2 E_{ox}}{\partial z^2} \right. \\ \left. - \cancel{\omega \frac{\partial^2 B_{ox}}{\partial z \partial y}} \right\} = 0.$$

(by applying (11a) & (11b))

$$\boxed{\checkmark \left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega^2}{v^2} - k_x^2 \right) \right] E_{ox} = 0.} \rightarrow (12a)$$

∇_T^2

(8)

* From page (10) \rightarrow .

Applying (11c) & (11d) in (2), $\vec{\nabla} \cdot \vec{B} = 0$. we can modify

Show that

$$\checkmark \left[\underbrace{\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}_{= \nabla^2} + \left(\frac{w^2}{v^2} - k_n^2 \right) \right] B_{0x} = 0 \rightarrow (12b)$$

on (2a) & (12b) \rightarrow
We don't set $w = v k$ as the walls

of the waveguide may impose a different dispersion relation. Using (12a) and (12b)
for E_{0x} & B_{0x} and applying them in (11a) - (11d), $E_{0y}, E_{0z}, B_{0y}, B_{0z}$ can also be obtained.

* If $E_{0x} = 0$, then that guided $\epsilon-m$ waves are said to

be Transverse electric (TE) waves (as longitudinal \vec{E} field is absent)

$$\left[\vec{B} = \vec{E}_L + \vec{E}_T = i \vec{E}_x + (j \vec{E}_y + k \vec{E}_z) \rightarrow \vec{E} = \vec{E}_T \right]$$

$\downarrow = 0$

* If $B_{0x} \neq 0$, then the waves are called transversed magnetic (TM) waves. ($\vec{B} \rightarrow \vec{B}_T$ as longitudinal \vec{B} field is absent)

* If $B_{0x} = 0, B_{0z} = 0$, the guided wave is said

to be transversed $\epsilon-m$ (TEM) wave.

$$(\vec{E} \rightarrow \vec{E}_T, \vec{B} \rightarrow \vec{B}_T).$$

But TEM waves cannot propagate inside an unshielded wave guide. (hollow conductor). It can be formed easily.

Menzel's eqn (1) $\rightarrow \vec{\nabla} \cdot \vec{E} = 0$.

$$\rightarrow j k_n B_{0x} + \frac{\partial E_{0y}}{\partial y} + \frac{\partial E_{0z}}{\partial z} = 0. \rightarrow \text{by } (1).$$

$$\because E_{0x} = 0 \rightarrow \frac{\partial E_{0y}}{\partial y} + \frac{\partial E_{0z}}{\partial z} = 0 \rightarrow (13a)$$

for TEM wave

\rightarrow go to page (10) **

(11)

** From page 8

$$\text{Maxwell's eqn } (3) \rightarrow (\vec{\nabla} \times \vec{B})_x = (j\omega \vec{B})_x$$

$$(9a) \rightarrow \frac{\partial B_{0z}}{\partial y} - \frac{\partial B_{0y}}{\partial z} = 0. \quad \left(B_{0x} = 0 \text{ for TEM wave} \right) \rightarrow (13b)$$

$$\frac{\partial}{\partial y} (13a) - \frac{\partial}{\partial z} (13b) \Rightarrow \frac{\partial^2 B_{0y}}{\partial y^2} + \frac{\partial^2 B_{0y}}{\partial z^2} = 0.$$

$$\Rightarrow \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) B_{0y} = 0 \\ \Rightarrow \boxed{\nabla_T^2 B_{0y} = 0.} \rightarrow (14).$$

Now by Maxwell's eqn (2) \rightarrow

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow jk_x B_{0x} + \frac{\partial B_{0y}}{\partial y} + \frac{\partial B_{0z}}{\partial z} = 0 \\ \Rightarrow \frac{\partial B_{0y}}{\partial y} + \frac{\partial B_{0z}}{\partial z} = 0 \quad \left(\because B_{0x} = 0 \text{ for TEM wave} \right) \rightarrow (15a).$$

by Maxwell's eqn (4) $\rightarrow (\vec{\nabla} \times \vec{B})_x = -j\mu\epsilon\omega \vec{E}_x.$

$$(14a) \rightarrow \frac{\partial B_{0z}}{\partial y} - \frac{\partial B_{0y}}{\partial z} = -j\mu\epsilon\omega E_{0x} = 0 \quad \left| \begin{array}{l} \because E_{0x} = 0 \\ \text{for TEM wave} \end{array} \right. \rightarrow (15b)$$

$$\frac{\partial}{\partial y} (15a) - \frac{\partial}{\partial z} (15b) \Rightarrow \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) B_{0y} = 0. \\ \Rightarrow \boxed{\nabla_T^2 B_{0y} = 0.} \rightarrow (16)$$

$$(9b), \Rightarrow k_x E_{0z} = \omega B_{0y} \quad \left(\because E_{0x} = 0. \right)$$

$$\Rightarrow \nabla_T^2 E_{0z} = \frac{\omega}{k_x} \nabla_T^2 B_{0y}$$

$$\therefore \boxed{\nabla_T^2 E_{0z} = 0} \quad \text{by eqn (16)} \\ \rightarrow (17).$$

(12)

$$(10b) \rightarrow +k_x B_{0z} = \mu \epsilon \omega E_{0y} \quad (\because B_{0x} = 0)$$

$$\Rightarrow \nabla_T^2 B_{0z} = \frac{\mu \epsilon \omega}{k_u} \nabla_T^2 E_{0y}$$

$$\therefore \boxed{\nabla_T^2 B_{0z} = 0} \rightarrow (18).$$

$$(14), (17) \rightarrow \nabla_T^2 (E_{0y} \hat{j} + E_{0z} \hat{k}) = 0.$$

$$\Rightarrow \boxed{\nabla_T^2 \vec{E} = 0.} \rightarrow (19) \quad (B_{0x} = 0) — \text{TEM wave}$$

In the same way, by eqns (16) & (18) \rightarrow

$$\boxed{\nabla_T^2 \vec{B} = 0.} \rightarrow (20) \quad (B_{0x} = 0) — \text{TEM Wave}$$

$$(15) \rightarrow \vec{E} = 0 = -\vec{\nabla} \phi, \quad (E_{0x} = 0)$$

$\therefore \phi = \text{constat.}, \text{i.e. the potential throughout}$
 $\text{is a constant and hence the electric field is zero inside}$
 $\text{the hollow pipe} — \text{so, no TEM wave at all inside}$
 $\text{the hollow pipe (within a hollow \del{wave} wave guide}$
 $\text{with perfectly conducting walls).}$

For a TEM mode to exist, we ^{may} ~~can~~ run
~~a~~ a separate conductor down the middle (e.g.
 a coaxial cable), i.e., TEM wave may occur in waveguides
 with a conducting core.

Rectangular wave guide -

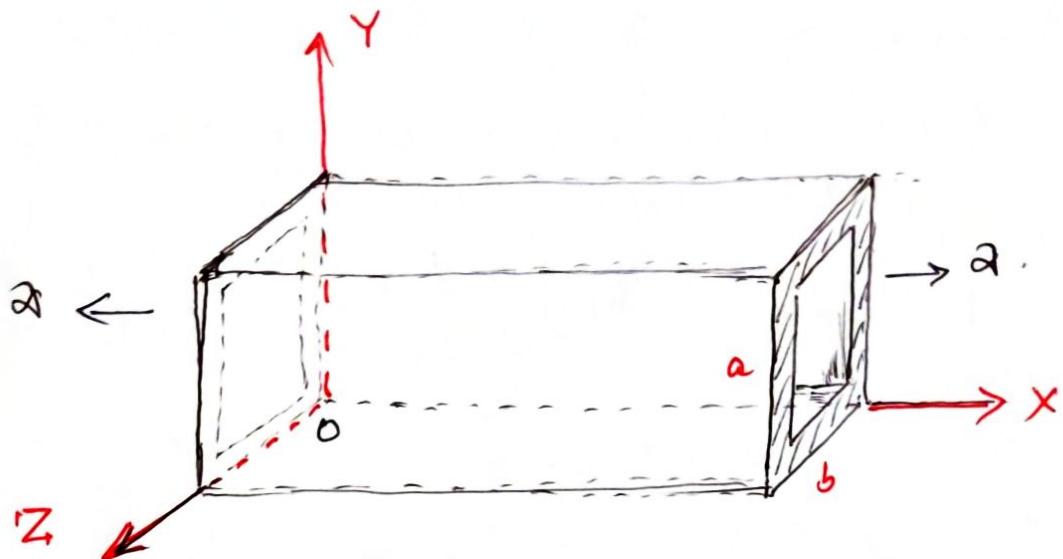


Fig - 2

We have a waveguide of rectangular shape (Fig-2), with height 'a' and width 'b' ($b < a$) and of infinite length with perfectly conducting walls. Since, the material of the wall is perfectly conductor, so, inside the material itself $\vec{E} = 0$ & $\vec{B} = 0$

and hence the boundary conditions is at the inner wall are -

$$\vec{E}_{||} = 0 \quad \text{and} \quad \vec{B}_{\perp} = 0 \quad \rightarrow ①.$$

$$\begin{aligned} \therefore \vec{E} &= \vec{E}_0 e^{j(k_x x - \omega t)} \\ \vec{B} &= \vec{B}_0 e^{j(k_x x - \omega t)} \end{aligned} \quad \left. \begin{array}{l} \text{Monochromatic guided} \\ \text{wave down the tube.} \end{array} \right\}$$

$$\vec{E} = \hat{i}E_x + \hat{j}E_y + \hat{k}E_z = \vec{E}_L + \vec{E}_T$$

$$\vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z = \vec{B}_L + \vec{B}_T$$

The subscripts L & T represents longitudinal (along the axis of the waveguide) and transversed

(14)

component (on the cross-section \perp to the axis of the waveguide). One B_x & B_z (longitudinal components) are determined by solving the differential eqns (subject to the boundary conditions)

$$\left[D_T^2 + \left(\frac{\omega^2}{v^2} - k_n^2 \right) \right] \begin{pmatrix} E_{0x} \\ B_{0x} \end{pmatrix} = 0, \quad \text{by eqns 12a \& 12b} \quad (12)$$

The all other components of \vec{E} and \vec{B} could be calculated out by using eqns in (11).

TE waves in rectangular waveguide -

$E_{0x} = 0 \rightarrow E_n = 0 (= E_L)$ (\vec{E}_T is present in the wave, so TE wave)

and the diff eqn for B_{0x} -

$$\left[D_T^2 + \left(\frac{\omega^2}{v^2} - k_n^2 \right) \right] B_{0x} = 0. \quad \rightarrow (22)$$

To solve (22), we proceed by the separation of variables with the ansatz

$$B_{0x}(y, z) = Y(y) Z(z). \quad \rightarrow (23)$$

$$(22) \rightarrow Z \frac{d^2 Y}{dy^2} + Y \frac{d^2 Z}{dz^2} + \left(\frac{\omega^2}{v^2} - k_n^2 \right) YZ = 0$$

$$\Rightarrow \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} + \left(\frac{\omega^2}{v^2} - k_n^2 \right) = 0.$$

The y and z dependent terms in the LHS must be each be constant:

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = -k_y^2 \quad \text{and} \quad \frac{1}{Z} \frac{d^2 Z}{dz^2} = -k_z^2$$

$\rightarrow (24) \qquad \qquad \qquad \rightarrow (25)$

(15)

subject to the constraint

$$-k_y^2 - k_z^2 + \left(\frac{\omega^2}{\nu\nu} - k_x^2 \right) = 0 \rightarrow (26)$$

The standard solutions of (24) & (25) →

$$\frac{d^2Y}{dy^2} + k_y^2 Y = 0 \rightarrow Y(y) = A \sin(k_y y) + B \cos(k_y y) \rightarrow (27)$$

$$\frac{d^2Z}{dz^2} + k_z^2 Z = 0 \rightarrow Z(z) = C \sin(k_z z) + D \cos(k_z z) \rightarrow (28)$$

The boundary conditions require -

$$\text{at } y=0 \text{ and } y=a, \quad B_y = 0 \quad \left(\because \vec{B}_L = 0 \right). \quad \left. \right\} (29)$$

$$\text{and at } z=0 \text{ and } z=b, \quad B_z = 0.$$

Using ~~(24)~~ and ~~(25)~~ (23), ~~(26)~~ in (11) →

$$(11a) \rightarrow B_{0y} = \frac{j}{\left(\frac{\omega^2}{\nu\nu} - k_x^2 \right)} \left(\omega Y \frac{dZ}{dz} \right) \quad (B_{0x} = 0). \quad \left. \right\}$$

$$(11b) \rightarrow B_{0z} = \frac{j}{\left(\frac{\omega^2}{\nu\nu} - k_x^2 \right)} \left(-\omega Z \frac{dY}{dy} \right) \quad \left. \right\} (30)$$

$$(11c) \rightarrow B_{0y} = \frac{j}{\left(\frac{\omega^2}{\nu\nu} - k_x^2 \right)} \left(k_x Y \frac{dZ}{dz} \right)$$

$$(11d) \rightarrow B_{0z} = \frac{j}{\left(\frac{\omega^2}{\nu\nu} - k_x^2 \right)} \left(k_x Y \frac{dZ}{dz} \right).$$

By eqn (29), boundary conditions; at the boundaries $y=0$ and $y=a$, $B_y = 0$, so, (30c) →

(16)

$$\frac{dy}{dy} = 0. \rightarrow (31)$$

$$\Rightarrow A k_y \cos(k_y y) - B k_y \sin(k_y y) = 0. \quad (\text{Applying } (27))$$

The above eqn is satisfied at $y=0$, only when $A=0$
 (if $A \neq 0$, then at $y=0$, $A k_y = 0 \Rightarrow k_y = 0$, not practical)

$$(27) \rightarrow Y(y) = B \cos(k_y y). \rightarrow (27')$$

Again by the boundary condition, at $y=a \rightarrow B \cos(k_y a) = 0$.

so, $\cancel{y(a)} = B \sin(k_y a) = 0$

$\therefore B \neq 0$, so $\sin(k_y a) = 0 \Rightarrow k_y a = m\pi$

(27') in (31) \rightarrow

$$B k_y \sin(k_y y) = 0.$$

$$\text{At, } y=a \rightarrow B k_y \sin(k_y a) = 0.$$

$$\Rightarrow \sin(k_y a) = 0. \Rightarrow \sin(m\pi), \quad m=0, \pm 1, \pm 2, \dots$$

$$\Rightarrow \left| k_y = \frac{m\pi}{a} \right| \rightarrow (32)$$

The same goes for Z , with $c=0$ and

$$Z(z) = D \cos(k_2 z), \quad \left| k_2 = \frac{n\pi}{b} \right| \rightarrow (33) \quad n=0, \pm 1, \pm 2, \dots$$

$$\therefore B_{ox}(y, z) = Y(y) Z(z) \quad \text{by (23)}.$$

$$= (B \cos(k_y y)) (D \cos(k_2 z)) \quad \text{by (27') \& (28')}$$

$$\boxed{B_{ox} = B_0 \cos\left(\frac{m\pi}{a}y\right) \cos\left(\frac{n\pi}{b}z\right)} \rightarrow (34)$$

(17)

Each pair (m, n) forms a perpendicular vibration of the B_{0x} field in the waveguide (eqn (22)), and these are called $TE_{m,n}$ modes. By convention the first index^(m) refers to the largest side of the rectangle (here a).

The wave number (k_x) of the TE_{mn} waves travelling inside the rectangular waveguide are found to be

$$k_x = \sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}} \quad \rightarrow (35)$$

If $\left(\frac{\omega}{c}\right)^2 < \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)$

or $\omega < \sqrt{c\pi \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)} \equiv \omega_{mn}$, $\rightarrow (36)$

the wavenumber k_x becomes imaginary and $(k_x = j\tilde{k}_x)$ and hence -

$$\vec{B} = \vec{B}_0 e^{j(j\tilde{k}_x x - \omega t)}$$

$$\therefore \vec{B} = \vec{B}_0 e^{-\tilde{k}_x x} e^{-j\omega t}$$

and so, instead of getting a travelling wave, we have exponentially attenuated fields and so the wave is unable to propagate strongly in the waveguide.

So, ω_{mn} is called the 'cut off' or 'threshold'

frequency for the particular mode (m, n) in question.

(18)

The lowest cut off frequencies for a given rectangular waveguide (height a , width b (a)) for the TE_{10} is \rightarrow .

$$(35) \rightarrow \omega_{10} = \frac{\omega\pi}{a} \quad m=1, n=0.$$

and for $TE_{01} \rightarrow$

$$\omega_{01} = \frac{\omega\pi}{b}. \quad (37)$$

$$(35), (36) \rightarrow k_x = \sqrt{\frac{\omega^2}{v^2} - \frac{N_{mn}^2}{v^2}} = \frac{\omega}{v} \left[1 - \left(\frac{\omega_{mn}}{\omega} \right)^2 \right]^{1/2}$$

$$\text{Phase velocity } v_p = \frac{\omega}{k_x} = \frac{\omega}{\left(1 - \omega_{mn}^2/\omega^2 \right)^{1/2}}$$

$$\therefore v_p > v.$$

If the ~~interior~~ interior of the waveguide is vacuum $v = c$ and phase velocity becomes greater than the speed of light in vacuum! (when ω is above the cut off frequency ω_{mn}). But the group velocity v_g is well behaved -

$$v_g = \left. \frac{d\omega}{dk_x} \right|_{\omega} = \frac{1}{\frac{dk_x}{d\omega}} \quad (37) \rightarrow \frac{dk_x}{d\omega} = \sqrt{\frac{\omega^2}{v^2} - \frac{N_{mn}^2}{v^2}} \times \frac{2\omega}{v^2}$$

$$= \frac{1}{v} \frac{1}{\sqrt{1 - \omega_{mn}^2/\omega^2}}$$

$$v_g^2 = c^2 \sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}} \quad (\text{core} \rightarrow \text{vacuum})$$

$$\underline{v_g < c.}$$

(19)

The propagation vector (wave vector) of the original plane wave -

$$\vec{k} = \hat{i} k_x + \hat{j} k_y + \hat{k} k_z$$

$$= \hat{i} k_x + \hat{j} \frac{m\pi}{a} + \hat{k} \frac{n\pi}{b}$$

and frequency -

$$\omega = v |\vec{k}| = v \sqrt{k_x^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad (37)$$

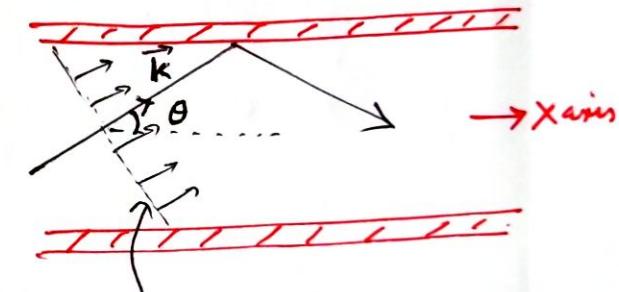
$$\boxed{\therefore \omega = v |\vec{k}| = \sqrt{v^2 k_x^2 + \omega_{mn}^2}} \quad (\text{by eqn } 36)$$

Only acute angles are allowed for the e-m wave, which is not travelling along the axis of the waveguide, but bounces off the surface of the waveguide.

$$\therefore k_x = |\vec{k}| \cos \theta$$

$$\therefore \cos \theta = \frac{k_x}{|\vec{k}|}$$

$$\boxed{\cos \theta = \sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}} \rightarrow (38)$$



Plane wave front

$$(37) \rightarrow \omega^2 = v^2 k^2 = v^2 k_x^2 + \omega_{mn}^2$$

$$v^2 = \omega^2 \frac{k_x^2}{k^2} + \frac{\omega_{mn}^2}{k^2}$$

$$\Rightarrow 1 = \frac{k_x^2}{k^2} + \frac{\omega_{mn}^2}{(vk)^2}$$

ω_{mn} and ω will determine the angle made by the propagation dirⁿ of e-m wave with the axis of the waveguide.

(20)

The em wave (plane wave) travels at the speed $v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ (or $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$), but since it is going at an angle θ with the axis of the wave guide, the act speed of the wave down the wave guide will be ~~be~~ is -

$$v_x = v \cos \theta = v \sqrt{1 - (\omega_{mn}/\omega)^2} \rightarrow (39).$$

? The TE₀₀ mode cannot occur in a rectangular wave guide. Why?

$$\text{If } m=0=n \rightarrow (35) \Rightarrow k_x = \sqrt{\frac{\omega^2}{a^2} - \frac{m^2 n^2}{a^2} - \frac{n^2 n^2}{b^2}}$$

$$\Rightarrow k_x = \frac{\omega}{v}$$

and as a result of this - by eqns (36) → $E_T (B_{0y}, E_{0z})$ and $B_T (B_{0y}, B_{0z})$ become ~~indeterminated~~ indeterminized, i.e. without any practical solutions, so, the TE₀₀ mode wave could not occur inside the waveguide of rectangular shaped cross-section.