CONCEPTS OF SPACE AND TIME IN EINSTEIN'S RELATIVITY

Einstein's Postulates of Relativity

According to the principle of Galilean Relativity, all mechanical laws remain symmetrical in all inertial frames and so all mechanical phenomena appear the same in all inertial frames. Einstein was convinced for general reasons that all physical laws including the laws of electromagnetism must be equally valid in all inertial frames of reference. So, all inertial frames are equally permissible to all physical laws and principles and therefore, they are equivalent (Principle of equivalence). Einstein raised this concept to the status of a postulate in his Relativity Theory, which states- ***"All physical laws and principles of nature are identical in all inertial frames of reference"***—(Einstein's Relativity Principle). This postulate elevates the Galileo's Relativity Principle of classical mechanics to the status of general law for the entire physics.

This above postulate therefore inherently rejects the need of any absolute or preferred/privileged frame like ether from the point of view of the applicability of natural laws. Since the Maxwell's laws of electromagnetism are the laws of nature and they are equally valid in all inertial frames, the speed of light in vacuum must be the same in all inertial frames, otherwise it would demand an specialized frame where the speed of light is equal to that value (c) obtained by Maxwell. Einstein followed this consequence of his above postulate very minutely and later on, he put forward this consequence in the form of another postulate of his Relativity Theory, which states- ***"the speed of light in vacuum is the same in all inertial frames, independent of the relative motion of observer and source of light"***---(Principle of Speed Constancy of Light)

The above two postulates are as a whole called Einstein's Special Relativity Theory.

Einstein's Relativity Principle still remains as a Postulate- Why?

The Relativity Principle has been verified experimentally to a very high degree of accuracy, but still it has been kept as a postulate, not as a law due to two reasons. All experimental instruments have their inherent limitations, so more sophisticated instruments invented in future may show deviation or defect of the principle. Again, there may be some undiscovered phenomena in nature which may not be within the realm of the principle and so we are not in a position to accept beforehand the principle for these phenomena.

Underlying meaning of the Principle of Einstein's Relativity:

According to the principle, all physical laws remain symmetrical in all inertial frames and so the phenomena appear the same in all inertial frames. If a piece of equipment in one inertial frame with a certain kind of machinery in it, the same machinery will work in another inertial frame in the way same as that in the former frame and therefore the conclusion of an experiment in both the frames is the same. That is the reason why we cannot distinguish one inertial frame from another one.

Space and time (from wiki)

The idea of time and space has occupied human thought for thousands of years. These things at first sight seem simple and easy to grasp, because they are close to everyday experience. Everything exists in time and space, so they appear as familiar conceptions. However, what is familiar is not necessarily understood. On closer examination, time and space are not so easily grasped. The dictionary is not much help here. Time is defined as a "a period," and a period is defined as "time." This does not get us very far! In reality, **the nature of time and space is quite a complex philosophical problem.**

It is common to say that time 'flows'. In fact, only material fluids can flow. Men and women clearly distinguish between past and future. A sense of time is, however, not unique to humans or even animals. Organisms often have a kind of "internal clock," like plants which turn one way during the day and another at night. **Time is an objective expression of the changing state of matter.** It is the way we express an actual process that exists in the physical world. Time is thus just an expression of the fact that all matter exists in a state of constant change. It is the destiny and necessity of all material things to change into something other than what they are and so time is inseparable from matter.

A sense of rhythm underlies everything: the heart-beat of a human, the rhythms of speech, the movement of the stars and planets, the rise and fall of the tides, the alternations of the seasons. These are deeply engraved upon the human consciousness, not as arbitrary imaginings, but as real phenomena expressing a profound truth about the universe. Time is a way of expressing change of state and motion which are inseparable features of matter in all its forms.

Space is the "otherness" of matter, to use Hegel's terminology, whereas time is the process whereby matter (and energy, which is the same thing) constantly changes into something other than what it is.

Space can also express change, as change of position. Matter exists and moves through space and the number of ways that this can occur is infinite: forward, backward, up or down, to any degree. There is a difference between time and space. Movement in space is reversible. Movement in time is irreversible. They are two different (and indeed contradictory) ways of expressing the same fundamental property of matter—change.

What about the concept of Space and Time in Einstein's Relativity

It was Einstein's conviction that Physical laws must be invariant and the speed of light in vacuum must be same equal to c in all frames and it would happen only at the cost of the Newtonian concept of absolute space and absolute time. So, Einstein reformulated the transformation equations for space and time coordinates in such a way that they always yield speed of light as a constant equal to c and space and time as two relative concepts.

According to Einstein, two people observing the same event in the same way could perceive the singular event occurring at two different times, depending upon their distance from the event in question. These types of differences arise from the time it takes for light to travel through space. Since light does travel at a finite and ever-constant speed, an observer from a more distant point will perceive an event as occurring later in time; however, the event is 'actually' occurring at the same instant in time. Thus, 'time' is dependent on space.

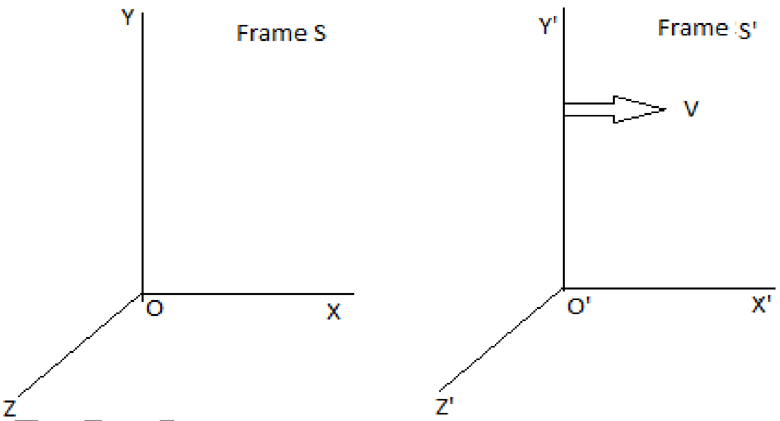
The same set of equations formulated by Einstein himself was also developed by Lorentz in the course of his mathematical study of electromagnetism. They derived the same set of equations from two different perspectives, so, to give due honour to both of them, the new transformation equations are called Lorentz-Einstein transformation equations.

Lorentz-Einstein Transformation Equations:

The postulate of Relativity Theory about the speed constancy of light in vacuum demands a new set of transformation equations other than that Galilean type, because those transformation demands that the speed never be an invariant. The new transformation equations must respect both the postulates of Einstein's Relativity and therefore they must satisfy the following four conditions-

1. The motion of a body in a straight line in one inertial frame S must be observed unaffected from another inertial frame S' moving with uniform velocity $\vec{V}$ relative to frame S.
2. If S' frame is moving with uniform velocity $\vec{V}$ relative to frame S, frame S will have a uniform velocity $-\vec{V}$ relative to frame S'.
3. The transformation themselves must satisfy the Einstein's Principle of Relativity.
4. The speed of light is the same in all inertial frames.

The first three conditions are beautifully satisfied by Galilean transformation equations. So, the form of the new set of transformation equations should have the form as below



$$x' = \alpha(x - Vt) \quad \text{.........................(1a)}$$

$$y' = y \quad \text{...................................(1b)}$$

$$z' = z \quad \text{...................................(1c)}$$

In the case where the $S'(x', y', z', t')$ inertial frame is moving with uniform speed V along +ve X-direction w. r. to $S(x, y, z, t)$ inertial frame and during the motion, the X'-axis coincides the X-axis, and Y'-axis remains parallel to Y-axis and Z'-axis to Z-axis.
The first three conditions demand an inverse transformation of the following type

$$x = \alpha\left(x' + Vt'\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2a)$$

$$y = y' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2b)$$

$$z = z' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2c)$$

Using equation (1a) in (2a), we have

$$x = \alpha\left[\alpha(x - Vt) + Vt'\right]$$

And then simplifying, we have

$$t' = \alpha\left[t + \left(\frac{1-\alpha^2}{\alpha^2}\right)\frac{x}{V}\right] \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

So, the new transformation equations will be

$$x' = \alpha\left(x - Vt\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4a)$$

$$y = y' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4b)$$

$$z = z' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4c)$$

$$t' = \alpha\left[t + \left(\frac{1-\alpha^2}{\alpha^2}\right)\frac{x}{V}\right] \ldots\ldots\ldots\ldots\ldots(4d)$$

If u_x and u'_x be the velocities of a moving particle w. r. to S and S' frames respectively, then by definition

$$u_x = \frac{dx}{dt} \qquad \text{and} \qquad u'_x = \frac{dx'}{dt'}$$

Hence

$$u'_x = \frac{dx'}{dt'} = \frac{\frac{dx'}{dt}}{\frac{dt'}{dt}} = \frac{\alpha\left(\frac{dx}{dt} - V\right)}{\alpha\left[1 + \left(\frac{1-\alpha^2}{\alpha^2}\right)\frac{1}{V}\frac{dx}{dt}\right]}$$

$$u'_x = \frac{u_x - V}{1 + \left(\frac{1-\alpha^2}{\alpha^2}\right)\frac{u_x}{V}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

(Using equations (4a) & (4d))

Instead of the material particle, if we imagine a moving light particle, i. e. a photon, then by the 4th condition, we have to write

$$u_x = u'_x = c$$

Hence from equation (5), we have

$$c = \frac{c - V}{1 + \left(\frac{1-\alpha^2}{\alpha^2}\right)\frac{c}{V}}$$

And then simplifying, we have

$$\alpha = \pm \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

We have to choose + ve sign for α, because the transformation equations in (4) only then **revert back to Galilean type in classical speed limit** $\left(\frac{V}{c} \ll 1\right)$. So, after putting the values of α in the new transformation equations in (4), we can readily obtain the Lorentz-Einstein transformation equations as follows

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} = \alpha(x - Vt) \;\ldots\ldots\ldots\text{(6a)}$$

$$y' = y \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(6b)}$$

$$z' = z \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(6c)}$$

$$t' = \frac{t - \frac{V}{c^2}x}{\sqrt{1 - \frac{V^2}{c^2}}} = \alpha(t - \frac{V}{c^2}x) \;\ldots\ldots\ldots\text{(6d)}$$

In matrix form

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & -\alpha V \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\alpha\frac{V}{c^2} & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \;\ldots\ldots\ldots\text{(7a)}$$

Or, $$x'_\mu = A_{\mu\nu} x_\nu\,, \qquad \mu,\nu = 1,2,3,4 \quad \text{…......……..(7b)}$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$ & $x_4 = t$ etc. and $A_{\mu\nu}$ is a 4 × 4 matrix.

And by substituting dashed coordinates by undashed ones and undashed by dashed and V by $-V$ in the above equations, we can easily derive the Inverse Lorentz-Einstein transformation equations as follows

$$x = \frac{x' + Vt'}{\sqrt{1-\frac{V^2}{c^2}}} \text{ ……………………….(8a)}$$

$$y = y' \text{ ……………………………….(8b)}$$

$$z = z' \text{ ……………………………….(8c)}$$

$$t = \frac{t' + \frac{V}{c^2}x}{\sqrt{1-\frac{V^2}{c^2}}} \text{ ……………………….(8d)}$$

** Since in **classical speed limit** $\left(\frac{V}{c} \ll 1\right)$, $1-\frac{V^2}{c^2} \approx 1$ and $\frac{V}{c^2} \approx 0$, hence the Lorentz- Einstein transformation equations revert back to Galilean type, i. e.

$$x' = x - Vt\,, \qquad y' = y\,, \qquad z' = z \quad \text{and} \quad t' = t$$

Concepts of Space and Time in Einstein's Relativity

The Lorentz-Einstein transformation equation for time shows that the time measurement of one observer in high speed comparable to the speed of light, there is mixed a little bit of space as seen by the other and in the same way, in the space measurement of one observer, a little bit of time of the other is mixed up. (But we cannot realize this mixing of space and time when we are moving with high speed.) And from it, we can logically come to the conclusion that for two different observers in two different frame moving with a relative velocity, both space and time for them never be identical and they therefore are relative.

Space and time are relative concept in Einstein's Relativity

In Einstein's relativity, one of the postulates is that the speed of light in free space is constant. If it would be so, time and space never remain as two absolute quantities irrespective of the motion of the observers. Let us assume that two events are occurred in S frame at two different points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ at the instants t_1 & t_2 respectively. Applying Lorentz-Einstein transformation equation for time, we can find out the time of occurrence of the events t'_1 & t'_2 w. r. to S' frame as follows.

$$t_1{'} = \frac{t_1 - \frac{V}{c^2}x_1}{\sqrt{1 - \frac{V^2}{c^2}}}, \qquad t_2{'} = \frac{t_2 - \frac{V}{c^2}x_2}{\sqrt{1 - \frac{V^2}{c^2}}}$$

It is clear from the above two expressions of t_1' and t_2' that if the events are simultaneous in S frame, but they are not appeared so from the S' frame. The meaning is that simultaneity is not an absolute concept, but it's a relative one.

Again, if someone is intending to measure the length of a moving rod, he has to calculate the difference of the readings taken simultaneously for the two ends. Since the simultaneity is a relative concept, the distance between the two ends will be different for different observer and therefore distance is a relative concept.

The conclusion now we can draw is that time and space are both relative concept and absolute space and absolute time are totally inadmissible in Einstein's relativity.

Simultaneity and order of events

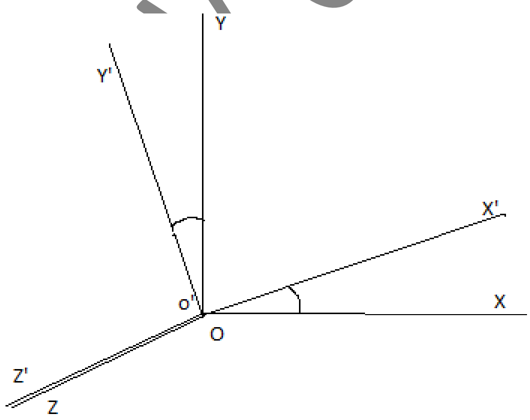
We suppose that two firecrackers explode simultaneously in S frame and these events both take place on X-axis, at $A(x_1, 0, 0, t_1)$ and $B(x_2, 0, 0, t_2)$ and $t_2' = t_1' = t$. Another observer in S' frame is also observing the events took place in S frame. The second observer has also recorded the positions and time of occurrence of the events as $A'(x_1', 0, 0, t_1')$ and $B'(x_2', 0, 0, t_2')$ from his own S' frame corresponding to the events occurred at $A(x_1, 0, 0, t_1)$ and $B(x_2, 0, 0, t_2)$ respectively. If we suppose that S' frame is moving with uniform speed V along +X axis w.r. to the inertial frame S, then applying the Lorentz-Einstein Transformation Equation for time coordinate, we have

$$t_1' = \frac{t_1 - \frac{V}{c^2}x_1}{\sqrt{1 - \frac{V^2}{c^2}}} \quad \text{and} \quad t_2' = \frac{t_2 - \frac{V}{c^2}x_2}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

And it is seen clearly that $t_2' = t'$ (not simultaneous) in S' frame even though $t_2' = t_1' = t$ in S frame and either $t_2' < t_1'$ or $t_2' > t'$, i.e. there will be an order of event w.r. to the observer in S' frame (i.e. any one of the two events will take place earlier than the other in S' frame). Thus, two events which are simultaneous in one inertial frame never be simultaneous w.r. to other inertial frames and there is an order of events for the other frames. It means that **simultaneity never be an absolute concept** as thought in Newtonian Mechanics (Galilean Relativity), but a relative one.

Since simultaneity is relative, space never be absolute. Let us take the example of a rod. The length of a rod can be determined correctly from the difference of the coordinates of the two ends of the rod. Since, simultaneity is relative, the distance between the two ends of the rod measured from one inertial frame never be the same as that from another inertial frame. Thus, **distance/space never be an absolute concept,** as thought in Newtonian Mechanics (Galilean Relativity), but a relative one.

Lorentz-Einstein transformation equations represent a rotation in coordinate axes.



Let the position coordinates of a point be x, y & z w. r. to the frame S and those w. r. to the frame S' be x', y' & z'. Here frame S' is the new position of the frame S after giving it a rotation through an angle θ about z axis. Thus

$$x' = x\,cos\theta + y\,sin\theta$$
$$y' = -x\,sin\theta + y\,cos\theta$$
$$z' = z$$

In matrix form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Or $\qquad x'_l = \boldsymbol{A}_{lm}\, x_m, \qquad l, m = 1,2,3$

(Summation is carried over repeated index according to **Einstein's summation convention**)

Where $\boldsymbol{A}$ is a 3X3 matrix and

$$x_1 = x, \quad x_2 = y, \ \& \quad x_3 = z$$

$$x'_1 = x', \quad x'_2 = y' \,\& \quad x'_3 = z'.$$

The Lorentz-Einstein transformation equations in (7a, 7b) are similar to those for rotation of coordinate system and that is why, it is loosely said that the Lorentz-Einstein transformation represents rotation in coordinate axes.

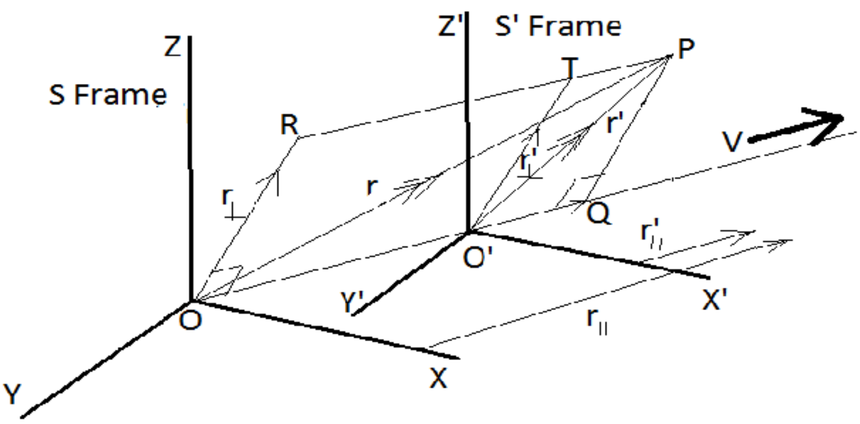
Lorentz-Einstein transformation for an arbitrary direction

The $S'\,(x', y', z')$. inertial frame is moving with uniform velocity $\boldsymbol{V}$ along any arbitrary direction w. r. to $S(x, y, z, t)$ inertial frame. We consider a position vector $\boldsymbol{r}$ for the point P w.r. to the origin O of S frame and resolving it parallel and perpendicular to the direction of motion of S' frame. From the above diagram

$$\boldsymbol{OP} = \boldsymbol{OQ} + \boldsymbol{OR}$$

Or

$$\boldsymbol{r} = \boldsymbol{r}_\parallel + \boldsymbol{r}_\perp \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$



We can express $\boldsymbol{r}_\parallel$ and $\boldsymbol{r}_\perp$ in terms of $\boldsymbol{r}$ as follows.

$$r_{\parallel} = \boldsymbol{r}.\boldsymbol{\hat{n}} \Rightarrow \boldsymbol{r}_{\parallel} = (\boldsymbol{r}.\boldsymbol{\hat{n}})\boldsymbol{\hat{n}} = \left(\frac{\boldsymbol{r}.\boldsymbol{V}}{V^2}\right)\boldsymbol{V} \text{..........(2a)}$$

($\boldsymbol{\hat{n}} = \frac{\boldsymbol{V}}{V}$ is the unit vector along the direction of propagation of the S' frame, i.e. along $\boldsymbol{V}$ **)**

$$\boldsymbol{r}_{\perp} = \boldsymbol{r} - \boldsymbol{r}_{\parallel} = \boldsymbol{r} - \left(\frac{\boldsymbol{r}.\boldsymbol{V}}{V^2}\right)\boldsymbol{V} \text{.....................(2b)}$$

Again, w. r. to the origin O' of S' frame, the position vector of the point P is $\boldsymbol{r'}$ and resolving it as above, we have

$$\boldsymbol{O'P} = \boldsymbol{O'Q} + \boldsymbol{O'T}$$

Or

$$\boldsymbol{r'} = \boldsymbol{r'}_{\parallel} + \boldsymbol{r'}_{\perp} \text{................................(3)}$$

By the Lorentz-Einstein transformation equations in (6)-

$$\boldsymbol{r'}_{\parallel} = \frac{\boldsymbol{r}_{\parallel} - \boldsymbol{V}t}{\sqrt{1-\frac{V^2}{c^2}}} \text{..................................(4a)}$$

$$\boldsymbol{r'}_{\perp} = \boldsymbol{r}_{\perp} \text{...............................(4b)}$$

(since $\boldsymbol{OT} = \boldsymbol{OR}$**)**

So, from equations (3), (4a) and (4b)

$$\boldsymbol{r'} = \boldsymbol{r'}_{\parallel} + \boldsymbol{r'}_{\perp}$$

And hence

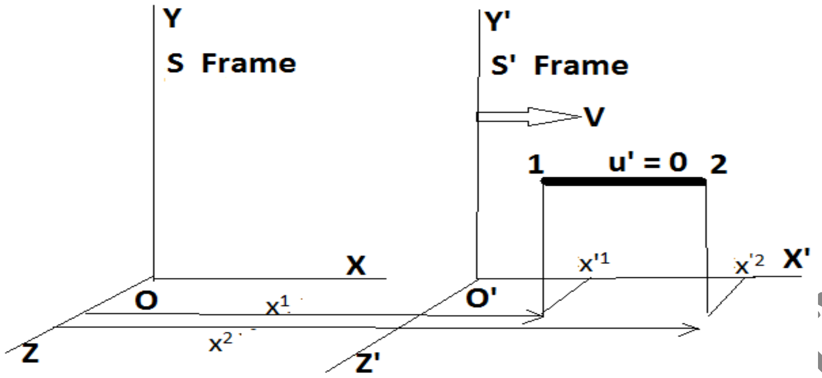
$$\boldsymbol{r'} = \frac{\boldsymbol{r}_{\parallel} - \boldsymbol{V}t}{\sqrt{1-\frac{V^2}{c^2}}} + \boldsymbol{r}_{\perp} = \frac{\left(\frac{\boldsymbol{r}.\boldsymbol{V}}{V^2}\right)\boldsymbol{V} - \boldsymbol{V}t}{\sqrt{1-\frac{V^2}{c^2}}} + \left(\boldsymbol{r} - \left(\frac{\boldsymbol{r}.\boldsymbol{V}}{V^2}\right)\boldsymbol{V}\right) \text{..................(5a)}$$

And by the Lorentz-Einstein transformation equation for time in equation (6d)

$$t' = \frac{t - \frac{V}{c^2}(OQ)}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{t - \frac{\boldsymbol{V}.\boldsymbol{r}}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} \text{...............(5b)}$$

These two equations (5a) and (5b) are the Lorentz-Einstein transformation equations for arbitrary direction of motion for S' frame w. r. to S frame.

Length Contraction



The length of a rod placed at rest parallel to X' axis in S' frame is measured by two observers stationed at the origins of S frame and S' frame. The observers at O and O' have to take the readings of the two ends 1 and 2 on the X and X' axes respectively.

For the observer in S' frame, the length, called **proper length**, is

$$L' = x'^2 - x'^1$$

And for the observer in S frame, the length is

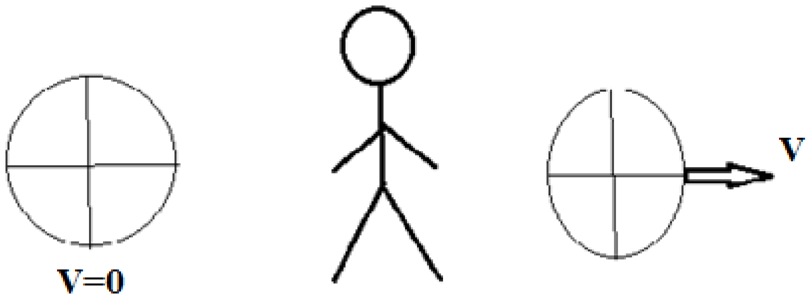
$$L = x^2 - x^1$$

But here it should be remembered that the rod is a moving one for the observer in S frame and so he has to note down the readings x^2 & x^1 of ends 1 and 2 simultaneously $t^1 = t^2$, otherwise the rod will change its position w. r. to the observer.

Now applying the Lorentz-Einstein transformation equation for space coordinate, it can be readily shown that

$$L = L'\sqrt{1 - \frac{V^2}{c^2}}$$

Since $V < c$, $\sqrt{1 - \frac{V^2}{c^2}} < 1$, the above relation shows that $L < L'$, i.e. the observed length from S frame from which the rod is in motion is found to be contracted by the factor $\sqrt{1 - \frac{V^2}{c^2}}$. But for a rod placed parallel to Y' or Z' axis, the length will not get contracted, as the coordinates corresponding to these two axes do not change during the motion of S' frame. So, we can conclude that the dimension of a moving object parallel to the direction of motion only gets contracted w. r. to the stationary observer.



- Since in classical speed limit$\left(\frac{V}{c} \ll 1\right)$, $1-\frac{V^2}{c^2} \approx 1$, and so $L = L'$ and the meaning of which is that the space can be assumed as an absolute physical quantity. It indirectly reveals the exactness of Newtonian mechanics in studying the dynamics of slowly moving bodies.
- If possible, we suppose that the speed of the rod is equal to or greater than that of light in free space w. r. to a stationery observer. In such case the factor $\sqrt{1-\frac{V^2}{c^2}}$ would be zero or imaginary, and so the situation would be unphysical. The situation would be physical only when the speed of the rod is less than c. So, nobody can move with speed beyond c except light. **Light is the limiting speed (ultimate speed) for all in Einstein's relativity.**

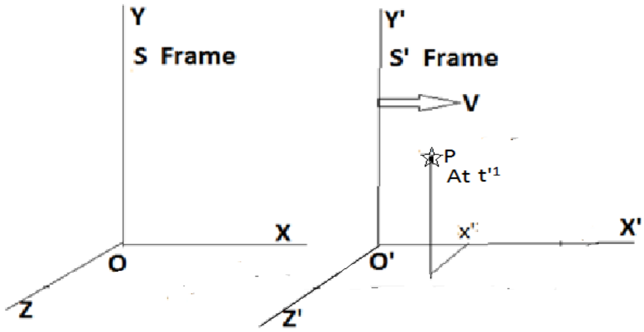
Time dilation or retardation of time

We suppose that two events occur at a point P (x') in S' inertial frame at the instant of time t'_1 & $t'_2(> t'_1)$ for the events 1 and 2 as registered by the clock at rest in S' frame moving with uniform speed V w.r. to S inertial frame. So, the time interval for the observer in S' frame is

$$\Delta t' = t'_2 - t'_1$$

which is called the proper or intrinsic time interval.

If the time of occurrence of these events as registered by a clock at rest in S frame be t_1 corresponding to event 1 and t_2 corresponding to the event 2, then the time interval for the observer in S frame is $\Delta t = t_2 - t_1$

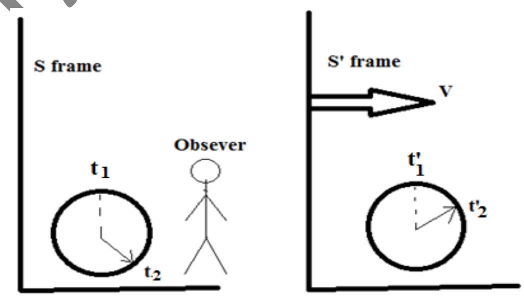


Using inverse Lorentz Einstein Transformation equation (8d) for time

$$\Delta t = \frac{t_2' + \frac{V}{c^2}x_2'}{\sqrt{1-\frac{V^2}{c^2}}} - \frac{t_1' + \frac{V}{c^2}x_1'}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Since $x_1' = x_2' = x'$, $\quad\therefore \Delta t = \frac{\Delta t'}{\sqrt{1-\frac{V^2}{c^2}}}$

Since $V < c$, $\sqrt{1-\frac{V^2}{c^2}} < 1$, the above relation shows that $\Delta t > \Delta t'$, i.e. the time interval registered by the clock from S frame is longer than that registered by another clock moving along with S' frame by the factor $\sqrt{1-\frac{V^2}{c^2}}$. So, the time gets dilated in S frame, whereas it gets retarded in S' frame w.r.t. the observer in S frame.



The clock (in S' frame) in motion with speed V w. r. to the S frame goes slow down by the factor $\sqrt{1-\frac{V^2}{c^2}}$, that is to say **a moving clock always goes slow w. r. to a stationary clock.** It can be said in a different way as every clock goes at its fastest rate when it is at rest w. r. to the observer and it goes on slowing down with its speed relative to the observer.

Experimental evidence for time dilation

Generally, the muons (Secondary Cosmic Ray (CR) Particles) are created at the top of the atmosphere, as the highly energetic Primary CR particles enter the atmosphere and hit the nuclei of air molecules. The muons just after creation start moving towards earth surface with a very high speed comparable to speed of light. The intensity of muons was first measured at the top of the mountain and then using an absorber with absorbing power equal to that of the air column from top of the mountain to the sea level, and again the intensity of muons was measured at the summit. The experimental results revealed that the intensity of muons at sea level was much lower than that of at the top of the mountain. The only possible explanation of the result is that these muons would be unstable particles and so they would undergo decay. The decay of muons can be described by the exponential law which states

$$I(t) = I_0 e^{-t/\tau},$$

Where I_0 and $I(t)$ are the intensities of muons at the beginning *t=0* before using the absorber and that of after time *t=t* when traversing through the absorber and τ is the mean life time. Using the measured intensities I_0 and $I(t)$ and putting the theoretical value of *t* in the above equation, the mean life time τ for muons is found out and it is found to be equal to $\tau = 10^{-5}$ s. To calculate *t*, we take $t = \frac{H}{V} \cong \frac{H}{c}$, where *H* is the distance from top of the mountain to sea level and *V* is the speed of muons through air and it is approximately equal to the speed of light *c* in free space.

According to Special Theory of Relativity, the life time τ for a moving muon w.r. to a stationary frame would be longer than that of τ_0 for a stationary muon (i.e. in the muon frame) and

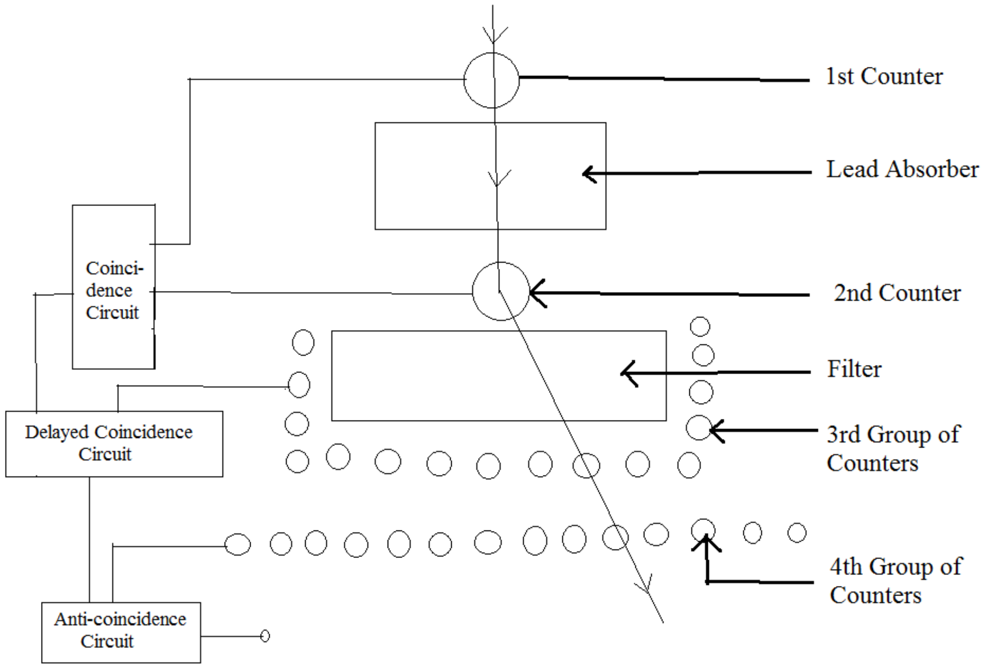
$$\tau = \frac{\tau_0}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

To calculate τ_0, we take the energy of a cosmic ray muon equal to $E = 10^9$ eV, which was found experimentally. From Einstein's mass energy equivalence principle

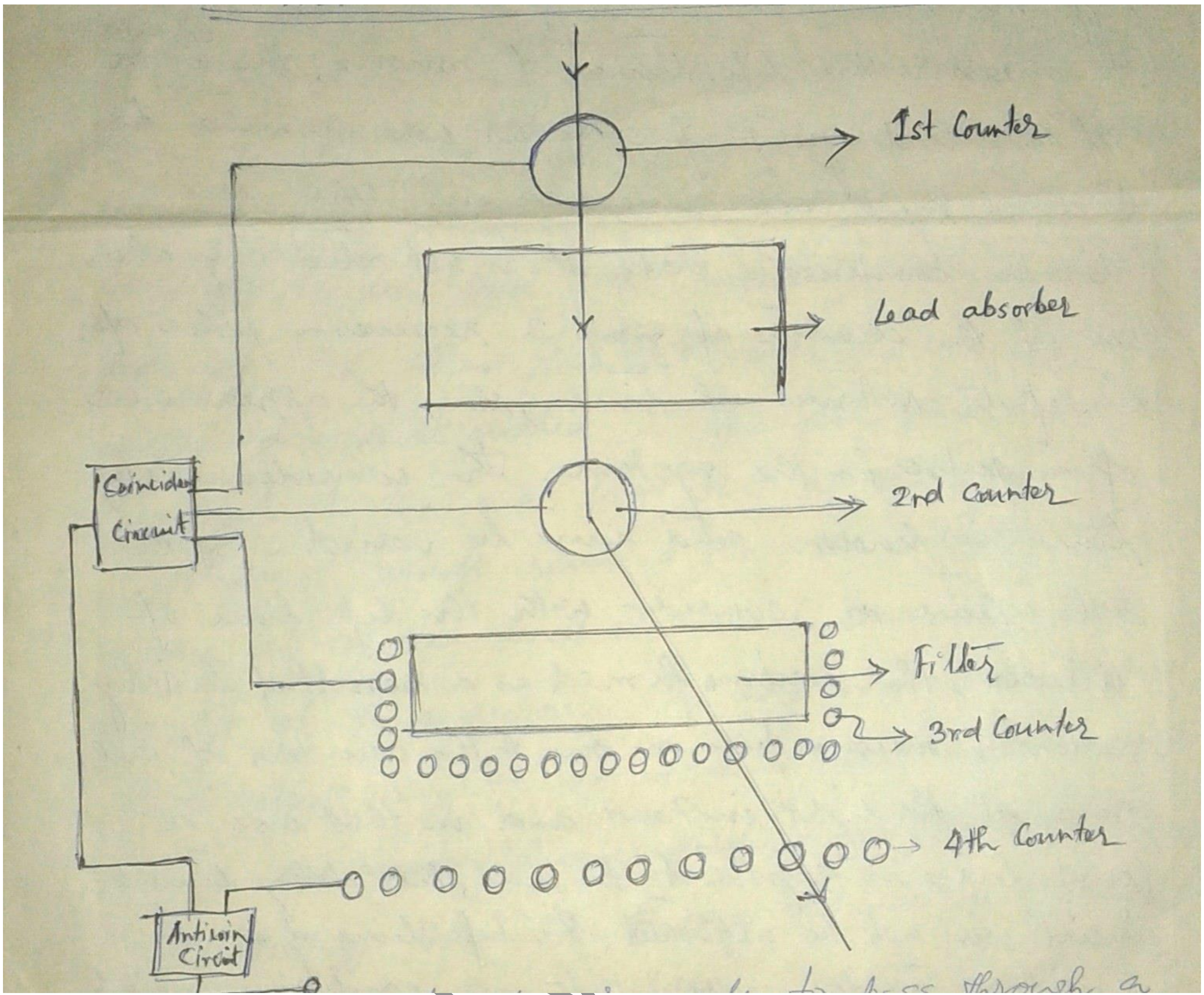
$$E = m_\mu c^2 = \frac{(m_\mu)_0 c^2}{\sqrt{1 - \frac{V^2}{c^2}}},$$

since rest mass energy of muons is $E_0 = (m_\mu)_0 c^2 = 10^8$ eV, so, $\sqrt{1 - \frac{V^2}{c^2}} = \frac{E_0}{E} = 0.1$. Now, considering the formula for time dilation, the mean life time of a stationary muon (in its own frame) is calculated out as $\tau_0 = 10^{-6} s = 1 \mu s$, which is shorter that the life time τ for a moving muon w.r. to the laboratory frame. If it can be shown experimentally that the life time is of the order of $1\mu s$, it will prove the validity of time dilation, otherwise this concept would have to be discarded.

Experimental set-up to determination of the mean life time of muons



CR particles are made to pass one by one through the 1^{st} counter, lead absorber for absorbing the secondary muons and 2^{nd} counter before entering the filter surrounded by the 3^{rd} group of GM counters and the 4^{th} group of counters below the filter. Both the 1^{st} and 2^{nd} counters are connected to a coincidence circuit whereas the 3^{rd} group of counters is connected with a delayed coincidence circuit and the 4^{th} group is connected with an anticoincidence circuit. The reason for connecting the 4^{th} group with an anticoincidence circuit is just to isolate the muons, that undergo disintegration inside the filter, from the other hard cosmic ray particles passing through the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} counters. In order to measure the life time of muons, the 3^{rd} group of GM counters is connected with a delayed coincidence circuit. The circuit has the characteristic that it is activated only when one of the counters in the 3^{rd} group in the surroundings of filter receives pulse after a definite interval of time since the appearance of a pulse in the 1^{st} and 2^{nd} counters. The coincidence delay time can be varied manually and that delayed time is known to the experimenters. If the delay time coincides with the life time of muons, the electrons formed as a result of disintegrating of muons fall on one of the counters of the 3^{rd} group at the right instant and in that case a $(\mu - e)$ decay is registered. If the delayed coincidence circuit is set for any other delayed time, the circuit will not be activated. So, by tuning the delayed time, the $(\mu - e)$ decay processes can be registered. The mean life time measured in this experiment gave conclusive evidence of time dilation as predicted by Special Relativity.


1st Counter
Lead absorber
Coincidence Circuit
2nd Counter
Filter
3rd Counter
4th Counter
Anticoin Circuit

Concept of Four Vector in Minkowski's spacetime

We consider an inertial frame $S'(x', y', z', t')$ moving with uniform speed V along +ve X-axis w.r. to inertial frame $S(x, y, z, t)$. By Lorentz-Einstein Transformation Equations (LETE)-

$$x' = \gamma(x - Vt) \qquad \text{(1a)}$$

$$y' = y \qquad \text{(1b)}$$

$$z' = z \qquad \text{(1c)}$$

$$t' = \gamma\left(t - \frac{V}{c^2}x\right) \qquad \text{(1d)}$$

Where $\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2}$.

Employing LETE

$$r'^2 - c^2t'^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \qquad \text{(in } S' \text{ frame)}$$

$$= \gamma^2(x - Vt)^2 + y^2 + z^2 - c^2\gamma^2\left(t - \frac{V}{c^2}x\right)^2$$

$$= \gamma^2\left[(x^2 + V^2t^2 - 2Vxt) - \left(c^2t^2 + \frac{V^2}{c^2}x^2 - 2Vxt\right)\right] + y^2 + z^2$$

$$= \gamma^2\left(1 - \frac{V^2}{c^2}\right)x^2 + \gamma^2(V^2 - c^2)t^2 + y^2 + z^2$$

$$= x^2 + \gamma^2\left(\frac{V^2}{c^2} - 1\right)c^2t^2 + y^2 + z^2$$

$$= x^2 + y^2 + z^2 - c^2t^2$$

Hence,

$$r'^2 - c^2t'^2 = r^2 - c^2t^2 \qquad \text{(2)}$$

That is, $x^2 + y^2 + z^2 - c^2t^2$ is a scalar invariant (Lorentz invariant) under LET.

Minkowski introduced the concept of **four dimensional spacetime continuum** with four coordinates $x_1,\ x_2,\ x_3\ \&\ x_4$, where $x_1 = x,\ x_2 = y,\ x_3 = z\ \&\ x_4 = ict\ (i^2 = -1)$. Here x_4 is kept imaginary for the fact that space and time essentially different and the factor c gives x_4 the same dimension as the other three space coordinates $x_1,\ x_2\ \&\ x_3$. With these new coordinates $x_1,\ x_2,\ x_3\ \&\ x_4$, the above Lorentz invariant can be written as

$$x^2 + y^2 + z^2 - c^2t^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \qquad \text{(In } S \text{ frame)}$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x'^2_1 + x'^2_2 + x'^2_3 + x'^2_4 \text{ (In } S' \text{ frame),}$$

And by eqn (2), they are equal

$$x'^2_1 + x'^2_2 + x'^2_3 + x'^2_4 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \text{.............................(3)}$$

So, $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is again called a Lorentz invariant and $x_1,\ x_2,\ x_3$ & x_4 are called the components of a true four dimensional position vector or **position four vector** x_μ.

$$x_\mu = (x_1,\ x_2,\ x_3,\ x_4) = (x,\ y,\ z,\ ict\ ) \text{....................................(4)}$$

${x_\mu}^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = x^2 + y^2 + z^2 - c^2t^2$ is called the norm of the position four vector x_μ. By eqn (3), this norm ${x_\mu}^2$ is a Lorentz invariant.

(Position vector in 3D space $\vec{r} = (x, y, z\ ) = (x_1,\ x_2,\ x_3)$ and norm $r^2 = x^2 + y^2 + z^2 = x_1^2 + x_2^2 + x_3^2$)

Now we are going to formulate TEs for the components of position four vector x_μ. For that, we have to call the LETEs from (1).

$$x' = \gamma(x - Vt) \rightarrow\ x'_1 = \gamma\left(x_1 - \frac{V}{ic}x_4\right)$$

$$\Rightarrow x'_1 = \gamma(x_1 + i\beta x_4) \text{ ....(5a)}$$

Here $\left(\beta = \frac{V}{c}\right)$

$$y' = y \rightarrow {x_2}' = x_2 \text{...(5b)}$$

$$z' = z \rightarrow {x_3}' = x_3 \text{ ....(5c)}$$

$$t' = \gamma\left(t - \frac{V}{c^2}x\right) \Rightarrow ict' = \gamma\left(ict - ic\frac{V}{c^2}x\right)$$

$$\Rightarrow x'_4 = \gamma(x_4 - i\beta x_1) \text{......................(5d)}$$

Or

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{...............................(5e)}$$

$$x'_i = \Omega_{ij}x_j \qquad (i, j = 1,2,3,4) \text{.......................(5f)}$$

(using Einstein's summation convention)

The above set of TEs (5a – 5d) are the TEs for the components of position four vector x_μ. Those TEs can be used to define a four vector.

Definition of four vectors / world vector

If the components of any four dimensional vector $A_\mu = (A_1,\ A_2,\ A_3,\ A_4)$ are transformed one inertial frame $S(x, y, z, t)$ to another $S'(x', y', z', t')$ moving with uniform speed V along +ve X-axis w.r. to inertial frame $S(x, y, z, t)$ in the fashion similar to that for position four vector x_μ as described in the eqns in (5) is known as a four vector and world vector.

Thus, the TEs or transformation rules for the four vector A_μ must be

$$A'_1 = \gamma(A_1 + i\beta A_4) \quad \text{..................(6a)}$$
$$A_2{}' = A_2 \text{..(6b)}$$
$$A_3{}' = A_3 \quad \text{.....................................(6c)}$$
$$A'_4 = \gamma(A - i\beta A_1) \text{......................(6d)}$$

Or

$$A'_i = \Omega_{ij} A_j \qquad (i, j = 1,2,3,4) \text{.......................(5f)}$$

Like the norm of position four vector x_μ, the norm of any four vector A_μ must also be Lorentz invariant.

Proof:

$$A_\mu{}'^2 = A_1{}'^2 + A_2{}'^2 + A_3{}'^2 + A_4{}'^2 \qquad \text{(in } S' \text{ frame)}$$
$$= \gamma^2(A_1 + i\beta A_4)^2 + A_2{}^2 + A_3{}^2 + \gamma^2(A_1 - i\beta A_4)^2$$
$$= \gamma^2 \begin{bmatrix} (A_1{}^2 - \beta^2 A_4{}^2 + 2i\beta A_1 A_4) + \\ (A_1{}^2 + \beta^2 A_4{}^2 - 2i\beta A_1 A_4) \end{bmatrix} + A_3{}^2 + A_4{}^2$$
$$= \gamma^2(1 - \beta^2)A_1{}^2 + \gamma^2(1 - \beta^2)A_4{}^2 + A_3{}^2 + A_4{}^2$$
$$= A_1{}^2 + A_2{}^2 + A_3{}^2 + A_4{}^2 = A_\mu{}^2 \qquad \text{(in } S \text{ frame)}$$

- The principle of relativity demands that all laws of nature must be invariant for all observers in the inertial frames. To ensure this invariantness of the laws, the physical quantities in terms of four vector notation rather than writing them in vector notation.

Some examples of four vectors

i) Position four vector

$$x_\mu = (x_1,\ x_2,\ x_3,\ x_4) = (x,\ y,\ z,\ ict\ )$$

Norm $x_\mu{}^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ is a Lorentz invariant.

Proof: $x'_\mu{}^2 = x'^2_1 + x'^2_2 + x'^2_3 + x'^2_4 \quad \rightarrow \quad x'_\mu{}^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_\mu{}^2$

HW-Apply the TEs from (5) and follow the same procedure as in the above proof for any arbitrary four vector.

ii) Displacement four vector

$$dx_\mu = (dx_1,\ dx_2,\ dx_3,\ dx_4) = (dx,\ dy,\ dz,\ icdt\ ) \qquad \text{(in differential form)}$$

Norm $\quad {dx_\mu}^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ is a Lorentz invariant.

Proof: $\quad {dx'_\mu}^2 = dx'^2_1 + dx'^2_2 + dx'^2_3 + dx'^2_4 \qquad$ (in S' frame)

$$= \gamma^2(dx_1 + i\beta dx_4)^2 + {dx_2}^2 + {dx_3}^2 + \gamma^2(dx_1 - i\beta dx_4)^2$$

$$=\gamma^2\begin{bmatrix}\left({dx_1}^2 - \beta^2{dx_4}^2 + 2i\beta dx_1 dx_4\right) + \\ \left({dx_1}^2 + \beta^2{dx_4}^2 - 2i\beta dx_1 dx_4\right)\end{bmatrix} + {dx_3}^2 + {dx_4}^2$$

$$= \gamma^2(1-\beta^2){dx_1}^2 + \gamma^2(1-\beta^2){dx_4}^2 + {dx_3}^2 + {dx_4}^2$$

$$= dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \qquad \text{(in } S \text{ frame)}$$

$$= {dx_\mu}^2$$

iii) Velocity four vector

We consider the uniform motion of a particle with velocity $\vec{v} = (v_x, v_y, v_z)$ in an inertial frame $S(x,y,z,t)$. The components of the velocity four vector $v_\mu = (v_1,\ v_2,\ v_3,\ v_4)$ in $S(x,y,z,t)$ are obtained by differentiating the corresponding components of the position four vector in $S(x,y,z,t)$ w.r.t. proper time τ in the frame attached to the moving particle, i.e.

$$v_\mu = \frac{dx_\mu}{d\tau} \qquad \text{here } \mu = 1,2,3,4$$

$$v_\mu = (v_1, \quad v_2, \quad v_3, \quad v_4)$$

$$v_\mu = \left(\frac{dx_1}{d\tau}, \quad \frac{dx_2}{d\tau}, \quad \frac{dx_3}{d\tau}, \quad \frac{dx_4}{d\tau}\right)$$

$$= \gamma\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \frac{d(ict)}{dt}\right)$$

Using Einstein's time dilation theorem, proper time τ in the frame attached to the moving particle is replaced with the time t of the laboratory frame $S(x,y,z,t)$.

$$dt = \frac{d\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma d\tau$$

$$v_\mu = \gamma(v_x, v_y, v_z, ic)$$

$$v_\mu = \gamma(\vec{v}, ic)$$

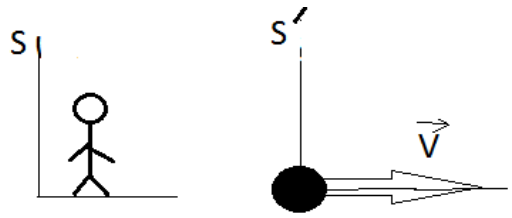
Norm $\quad {v_\mu}^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2$

$$= (\gamma v_x)^2 + \left(\gamma v_y\right)^2 + (\gamma v_z)^2 + (i\gamma c)^2$$

$= \gamma^2(v^2 - c^2) = -c^2$= always a constant independent of reference frames,

So, the norm of v_μ is a Lorentz invariant.

Alternately,



The velocity four vector for the particle in its own frame

$$v'_\mu = \frac{dx'_\mu}{d\tau} \qquad \text{here} \quad \mu = 1,2,3,4$$

$$v'_\mu = (v'_1, \quad v'_2, \quad v'_3, \quad v'_4)$$

$$v'_\mu = \left(\frac{dx'_1}{d\tau}, \quad \frac{dx'_2}{d\tau}, \quad \frac{dx'_3}{d\tau}, \quad \frac{dx'_4}{d\tau}\right)$$

Since, $dx'_1 = dx'_2 = dx'_3 = 0$ $(dx' = dy' = dz' = 0)$ and $dx'_4 = icdt' = icd\tau$,

$$v'_\mu = \gamma(0,0,0,ic) = \gamma\left(\vec{0}, ic\right)$$

Norm of v'_μ will be

$${v'_\mu}^2 = v'^2_1 + v'^2_2 + v'^2_3 + v'^2_4 = 0 + 0 + 0 + (ic)^2 = -c^2 \text{ (Same as norm of } v_\mu\text{)}$$

For both frames, the norm is the same.

iv) Momentum-energy four vector or four momentum

The components of this four vector is the product of the components of velocity four vector with the rest mass of the particle.

$$p_\mu = m_0 v_\mu = m_0\gamma(\vec{v}, ic) = m(\vec{v}, ic) = (m\vec{v}, imc)$$

Thus $\quad p_\mu = \left(\vec{p}, i\frac{E}{c}\right),$

where the relativistic mass of the particle $m = m_0\gamma$, relativistic momentum $\vec{p} = m\vec{v}$ and total energy $E = mc^2$.

Norm $\quad p_\mu{}^2 = \vec{p}^2 + \left(i\frac{E}{c}\right)^2 \qquad$ Applying $(E^2 = p^2c^2 + m_0^2c^4)$

$= p^2 - \left(\frac{E}{c}\right)^2 = -m_0c^2$= always a constant independent of reference frames,

So, the norm of p_μ is a Lorentz invariant.

- Since linear momentum and energy are coupled in the momentum-energy four vector in Minkowski's formalism, the conservation laws of linear momentum and energy appear here as a single law: law of conservation of four-momentum.

v) Force four vector or four-force or Minkowski force

The components of force four vector are obtained by differentiating the momentum-energy four vector w.r.t. the proper time τ, i.e.

$$F_\mu = \frac{dp_\mu}{d\tau} \qquad \text{here } \mu = 1,2,3,4,$$

which is the eqn of motion in Minkowski spacetime.

$$F_\mu = \frac{dp_\mu}{d\tau} = \gamma\frac{dp_\mu}{dt} = \gamma\left(\frac{d\vec{p}}{dt}, \frac{i}{c}\frac{dE}{dt}\right)$$

Hence, $$F_\mu = \gamma\left(\vec{F}, \frac{i}{c}\vec{F}.\vec{v}\right)$$

(Applying force vector $\vec{F} = \frac{d\vec{p}}{dt}$ and power $\frac{dE}{dt} = \vec{F}.\vec{v}$)

Norm $\quad F_\mu{}^2 = \gamma^2\left[\vec{F}^2 + \left(\frac{i}{c}\vec{F}.\vec{v}\right)^2\right]$

$= \gamma^2\left[F^2 - \frac{1}{c^2}\left(\vec{F}.\vec{v}\right)^2\right]$

$= \gamma^2\left(1 - \frac{v^2}{c^2}\right)F^2 = F^2$= always a constant independent of reference frames.

$\vec{F}$ is the Newtonian force considering absolute time w.r.t. all inertial frame and in that case, the force vector becomes an invariant under Galilean transformation. So, the norm of F_μ is a Lorentz invariant.

Transformation properties (equations) of momentum

The motion of a body (of rest mass m_0) moving with velocity $\vec{u'}$ in S' frame is observed by two observers from S' frame itself and from another inertial frame S. Let S' frame is moving with uniform speed V along $+X$ direction w.r. to S frame. During the motion of frame S', X' axis remains coincident with X axis and Y' remains parallel with Y and Z' axis with Z axis. If $\vec{u}$ be the velocity of the body as observed by the observer from S frame and if m be its relativistic mass in S frame and m' be the relativistic mass for the same body moving with velocity $\vec{u'}$ in S' frame, then by the mass variation theorem in Einstein's Relativity

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \qquad \text{and} \qquad m' = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}, \text{.....................(1)}$$

where $\vec{u} = \hat{i}u_x + \hat{j}u_y + \hat{k}u_z \qquad u^2 = u_x^2 + u_y^2 + u_z^2$

$$\vec{u'} = \hat{i}u'_x + \hat{j}u'_y + \hat{k}u'_z \qquad u'^2 = u_x'^2 + u_y'^2 + u_z'^2$$

The momenta for the body w.r. to S and S' frames

$$\vec{p} = \left(p_x, p_y, p_z\right) \qquad \text{and} \qquad \vec{p'} = \left(p'_x, p'_y, p'_z\right),$$

where $p_x = mu_x = \dfrac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}u_x, \qquad p_y = mu_y = \dfrac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}u_y, \qquad p_z = mu_z = \dfrac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}u_z,$

...................................(2)

and

$$p'_x = mu'_x = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}u'_x \qquad p'_y = mu'_y = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}u'_y \qquad p'_z = mu'_z = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}u'_z.$$

By applying the Lorentz-Einstein transformation equations

$$x' = \frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad y' = y, \qquad z' = z, \qquad t' = \frac{t - \frac{V}{c^2}x}{\sqrt{1-\frac{V^2}{c^2}}},$$

and also applying the definition of velocity, i.e. $u'_x = \dfrac{dx'}{dt'} = \dfrac{\frac{dx'}{dt}}{\frac{dt'}{dt}}$,etc., we can derive the following velocity addition theorem in Einstein's Relativity.

$$u'_x = \frac{u_x - V}{1 - \frac{V}{c^2}u_x}, \qquad u'_y = \left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}u_x}\right)u_y, \qquad u'_z = \left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}u_x}\right)u_z.$$

Thus, the components of momentum vector $\overrightarrow{p'}$ for the moving body in S' frame are

$$p'_x = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}\frac{u_x - V}{1-\frac{V}{c^2}u_x}, \qquad p'_y = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}\left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}u_x}\right)u_y, \qquad p'_z = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}\left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}u_x}\right)u_z. \qquad \text{..............................(3)}$$

We are now going to replace $\sqrt{1-\frac{u'^2}{c^2}}$ appeared in RHS in the above equations with an expression in S frame. To do it, we are exercising the following mathematics.

$$c^2 - u'^2 = c^2 - \left(u'^2_x + u'^2_y + u'^2_z\right)$$

$$= c^2 - \left[\left(\frac{u_x - V}{1-\frac{V}{c^2}u_x}\right)^2 + \left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}u_x}u_y\right)^2 + \left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}u_x}u_z\right)^2\right]$$

$$= c^2 - \frac{1}{\left(1-\frac{V}{c^2}u_x\right)^2}\left[\left(u_x - V\right)^2 + \left(1-\frac{V^2}{c^2}\right)u_y^2 + \left(1-\frac{V^2}{c^2}\right)u_z^2\right]$$

$$= c^2 - \frac{1}{\left(1-\frac{V}{c^2}u_x\right)^2}\left[u^2 + V^2 - 2u_xV - \frac{V^2}{c^2}\left(u_y^2 + u_z^2\right)\right]$$

$$= \frac{1}{\left(1-\frac{V}{c^2}u_x\right)^2}\left[c^2\left(1-\frac{V}{c^2}u_x\right)^2 - \left\{u^2 + V^2 - 2u_xV - \frac{V^2}{c^2}\left(u^2 - u_x^2\right)\right\}\right]$$

After simplifying

$$1-\frac{u'^2}{c^2}=\frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V}{c^2}u_x\right)^2}\left(1-\frac{u^2}{c^2}\right)$$

Applying it along with the mass-energy equivalence principle $E=mc^2$, mass variation formula $m=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$ from (1) and expressions of the components p_x, p_y, p_z from (2) in the three expressions for components p'_x, p'_y, p'_z in (3), we have

$$p'_x=\frac{p_x-\frac{V}{c^2}E}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad p'_y=p_y, \qquad p'_z=p_z$$

The inverse of these transformation relations can be obtained easily by changing the primed by unprimed and unprimed by primed quantities and V by $-V$, i.e. $\left(\vec{p},E\right) \rightarrow \left(\vec{p'},E'\right)$ and $V \rightarrow -V$.

$$p_x=\frac{p'_x+\frac{V}{c^2}E'}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad p_y=p'_y, \qquad p_z=p'_z.$$

Transformation properties (equations) for Energy

The motion of a body (of rest mass m_0) moving with velocity $\vec{u'}$ in S' frame is observed by two observers from S' frame itself and from another inertial frame S. Let S' frame is moving with uniform speed V along $+X$ direction w.r. to S frame. During the motion of frame S', X' axis remains coincident with X axis and Y' remains parallel with Y and Z' axis with Z axis. If $\vec{u}$ be the velocity of the body as observed by the observer from S frame and if m be its relativistic mass in S frame and m' be the relativistic mass for the same body moving with velocity $\vec{u'}$ in S' frame, then by the mass variation theorem in Einstein's Relativity

$$m=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \qquad \text{and} \qquad m'=\frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}},$$

where $\vec{u}=\hat{i}u_x+\hat{j}u_y+\hat{k}u_z \qquad u^2=u_x^2+u_y^2+u_z^2$

$$\overrightarrow{u'} = \hat{i}u'_x + \hat{j}u'_y + \hat{k}u'_z \qquad u'^2 = u'^2_x + u'^2_y + u'^2_z$$

By the mass-energy equivalence principle $E = mc^2$, the energy of the body w.r. to frame S and that of the body w.r. to frame S' are

$$E = mc^2 = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}c^2 \qquad E' = m'c^2 = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}c^2$$

Hence, energy transformation equation

$$E' = \frac{\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u'^2}{c^2}}}E\,.$$

We are now going to replace $\sqrt{1-\frac{u'^2}{c^2}}$ appeared in RHS in the above equations with an expression in S frame. To do it, we are exercising the following mathematics.

(same as the previous topic already discussed)

We have arrived at

$$1-\frac{u'^2}{c^2} = \frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V}{c^2}u_x\right)^2}\left(1-\frac{u^2}{c^2}\right)$$

Applying it in energy transformation relation

$$E' = \left(\frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V}{c^2}u_x\right)^2}\left(1-\frac{u^2}{c^2}\right)\right)^{-\frac{1}{2}}\sqrt{1-\frac{u^2}{c^2}}E \;= \frac{\left(1-\frac{V}{c^2}u_x\right)}{\sqrt{1-\frac{V^2}{c^2}}}E$$

Using $E = mc^2$ and $p_x = mu_x$, the final transformation relation for energy is obtained as below.

$$E' = \frac{E - p_x V}{\sqrt{1-\frac{V^2}{c^2}}}$$

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Relativistic dynamics, Relativistic Mass

In Newtonian Mechanics, the mass of a body is assumed to be a constant physical quantity independent of its speed w. r. to observers. By Newton's second law of motion, under the action of a constant force, the body moves with constant acceleration and if the body is constantly acted upon by the force for a longer time, the body keeps picking up of velocity and could finally move with speed greater than the speed of light c in free space. This goes vehemently against the finding in the Special Relativity Theory that the speed c is the natural upper limit for all objects. This is happening only when the influence of the force would gradually become less and less as the speed approaches c and would ultimately vanish. This view leads to the increase of inertia with speed, tending to infinity as the speed of the body approaches c and there would be practically no acceleration in the sense of velocity change. Since inertia is proportional to mass (larger the mass, larger is the inertia), the mass has to be increased with speed to infinity as speed approaches c .

The above conclusion can be mathematically stated as follows-

$$m_u = m_0 f(u)$$

where $m_0 =$ mass of the body at rest $(u=0)$, called **rest mass**

$m_u =$ mass of the body moving with speed u, called **relativistic mass**

$f(u) \to$ a function of speed u of the body such that

$f(u) \to 1$ as $u \to 0$

and $\to \infty$ as $u \to c$

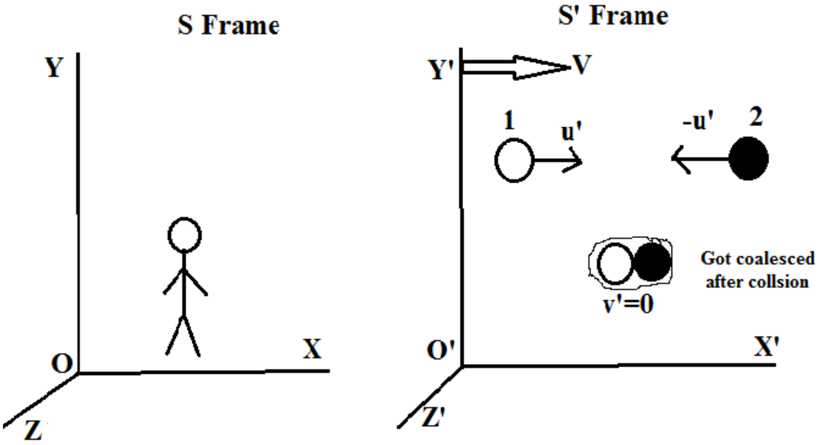
In 1904, Lorentz for the first time gave the **mass variation formula** as

$$m_u = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$

Derivation of mass variation formula

According to the Relativity Principle, the physical laws and principles are invariant in all inertial frames and same is the case for principle of conservation of linear momentum. If we suppose that Newtonian concept about constancy of mass with speed is correct, the said conservation principle valid in one inertial frame is found to be invalid in another inertial frame under Lorentz Einstein transformation. It simply indicates that the Newtonian concept of absolute mass is totally wrong, actually **mass is a relative concept**. It can be theoretically shown by studying a dynamical collision problem from two different inertial frames.

We consider a perfectly elastic collision process between two identical bodies 1 and 2 of mass $m^/$ moving with equal speed $u^/$ in opposite directions in $S^/$ inertial frame moving with uniform speed V along +X-direction w. r. to another inertial frame S as shown in the figure. After collision, they get coalesced and comes to rest in $S^/$ frame.



The observer in S frame is observing the same collision process and according to him, body 1 of mass m_1 and body 2 of mass m_2 are moving in opposite directions with the speed u_1 and u_2 respectively and after collision, they get stuck together and start moving with speed V along +X-direction. By the relativistic velocity addition formula, we can write-

$$u_1 = \frac{u' + \mathrm{V}}{1 + \dfrac{V}{c^2} u'} \quad \text{and} \quad u_2 = \frac{-u' + \mathrm{V}}{1 + \dfrac{V}{c^2}(-u')}$$

Applying the principle of conservation of linear momentum in the collision process as observed from the S frame, we have

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) V$$

Now putting the expressions for u_1 and u_2 and then simplifying, we have the mass ratio as

$$\frac{m_1}{m_2}=\frac{1+\frac{V}{c^2}u'}{1-\frac{V}{c^2}u'}\,.$$

To express this mass ratio in terms of u_1 and u_2 in S frame, we must evaluate $\frac{V}{c^2}u'$ using the above velocity formulae. Since

$$u_1^{\,2}\left(1+\frac{V}{c^2}u'\right)^2-u_2^{\,2}\left(1-\frac{V}{c^2}u'\right)^2=4u'\mathrm{V}\,,$$

we can mould it to the form of a quadratic equation of $\frac{V}{c^2}u'$ as follows

$$\frac{Vu'}{c^2}=\frac{-2\left(u_1^{\,2}+u_2^{\,2}-2c^2\right)\pm\sqrt{\left\{2\left(u_1^{\,2}+u_2^{\,2}-2c^2\right)\right\}^2-4\left(u_1^{\,2}-u_2^{\,2}\right)\left(u_1^{\,2}-u_2^{\,2}\right)}}{2\left(u_1^{\,2}-u_2^{\,2}\right)}\,.$$

Since no material body can move with the speed equal to c, $\frac{Vu'}{c^2}<1$ and therefore the –ve sign is taken into consideration. After simplification, we have

$$\frac{Vu'}{c^2}=\frac{2c^2-u_1^{\,2}-u_2^{\,2}-2\sqrt{\left(c^2-u_1^{\,2}\right)\left(c^2-u_2^{\,2}\right)}}{\left(u_1^{\,2}-u_2^{\,2}\right)}\,.$$

Thus,

$$\frac{m_1}{m_2}=\frac{\sqrt{1-\frac{u_2^{\,2}}{c^2}}}{\sqrt{1-\frac{u_1^{\,2}}{c^2}}}\,.$$

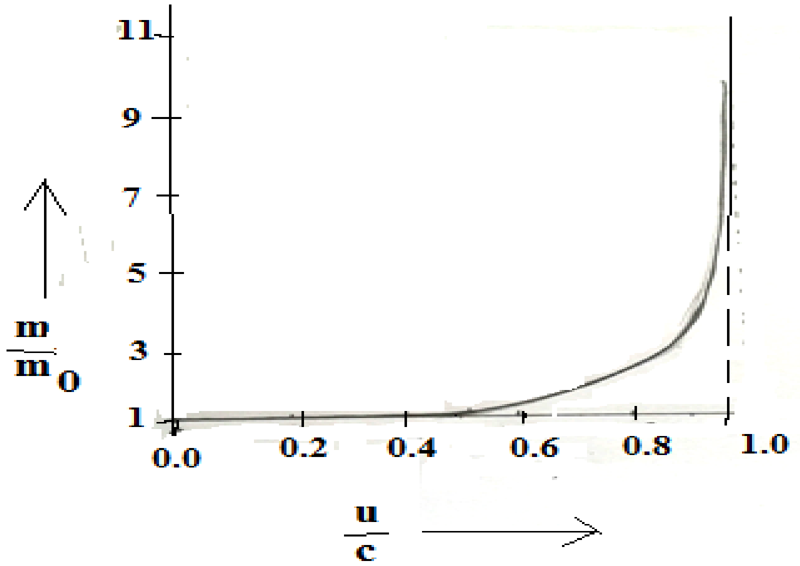
We suppose that that the body 2 was at rest before collision w.r. to the observer in S frame. So, putting $u_2=0$ and $m_2=m_0$ (which is again the rest mass for the body 1) in the above equation, the mass ratio reduces to the form

$$\frac{m_1}{m_0} = \frac{1}{\sqrt{1-\frac{u_1^{\,2}}{c^2}}}.$$

Dropping the index 1 from the above equation, we can make it more general for any object of relativistic mass m moving with the speed u relative to a stationary observer as

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}.$$

- The variation of mass with speed of a material particle becomes quite significant at high values of u. It can be shown by drawing a theoretical graph of $\frac{m}{m_0}$ against $\frac{u}{c}$ (since $\frac{m}{m_0} = \left[1-\left(\frac{u}{c}\right)^2\right]^{-1/2}$).



- The variation of mass with speed was first confirmed by Bucherer (1908), when he observed that the ratio of charge to the mass, i.e. $\frac{e}{m}$ of an electron is smaller for fast moving electrons than that for the slow moving electrons. It indicates the increase of mass with speed of the electrons as the charge is an invariant quantity in Relativistic Electrodynamics.

F_x.

Measured value of $\frac{u}{c}$	Measured value of $\frac{e}{m}$ in C/kg
0.3173	1.661 X 10^{11}
0.3787	1.630 X 10^{11}
0.4281	1.590 X 10^{11}
0.5154	1.511 X 10^{11}
0.6870	1.283 X 10^{11}

Working Principle of the experiment:

The magnetic force F_m exerted on a particle of charge q projected normally into a uniform magnetic field of magnetic flux density B provides the necessary centripetal force F_c to revolve in a circular path of radius r. Thus

$$F_m = F_c \Rightarrow quB = \frac{mu^2}{r}$$

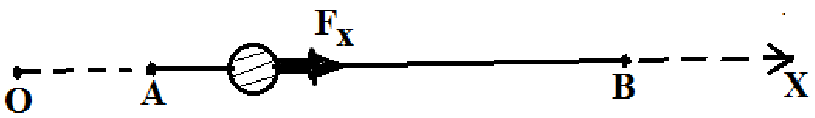
$$\therefore \quad \frac{q}{m} = \frac{u}{rB}$$

The electrons produced by any source (e.g. radioactive beta decay) are made to pass through a velocity selector so that a collimated beam of electrons moving with certain speed can be obtained. Those electrons are injected normally into a uniform magnetic field

- The **fine structure of Hydrogen spectrum** could be well explained only when relativistic mass of the electrons revolving round the nucleus is taken into consideration along with some other factors.

Mass-energy equivalence principle

In classical mechanics, the kinetic energy (KE) gained by a moving body is equal to the work done on the body. It can be proved very easily by considering the displacement of a body of constant mass m from A to B along +X-direction in space under the action of a variable force F_x.



$$W=\int_A^B dW=\int_A^B F_x\ dx$$

$$=\int_A^B \frac{d}{dt}(mu)\frac{dx}{dt}dt$$

$$=m\int_{u_A}^{u_B} u\ du$$

(since m is assumed to be an absolute quantity in Classical Physics)

$$=\frac{m}{2}\int_{u_A}^{u_B} d\left(u^2\right)=\frac{m}{2}\left[u^2\right]_{u_A}^{u_B}$$

$$=\frac{1}{2}m{u_B}^2-\frac{1}{2}m{u_A}^2$$

$\therefore$ Work done $W=(KE)_B-(KE)_A$

If work is done on the body (i.e. for +ve work done), the body will gain KE and if the body itself does work against a force (i.e. for -ve work done), the body will lose KE. Physically also it is true, but there is a problem in the mathematical calculation under the light of relativity. As the body is moving under the action of force, magnitude of the velocity of the body is changing with time and so mass of the body never be a constant and so it could not be brought out of the integration. So, we again repeat the same calculation considering mass variation formula as follows.

Change in KE $(\Delta K)=(KE)_B-(KE)_A=$ Work done (W)

$$=\int_A^B dW=\int_A^B F_x\ dx$$

$$=\int_A^B \frac{d}{dt}(mu)\frac{dx}{dt}dt$$

$$= \int_A^B u \; d(mu)$$

$$= \int_A^B u(u \; dm + m \; du)$$

$$\therefore \Delta K = (KE)_B - (KE)_A = \int_A^B (u^2 \; dm + mu \; du)$$

Using the mass variation formula

$$m^2c^2 - m^2u^2 = m_0^2c^2$$

And taking the total differential in both sides of the above equation and then simplifying, we have

$$u^2 \; dm + mu \; du = c^2 \; dm \, .$$

$$\therefore \Delta K = (KE)_B - (KE)_A = \int_{m_A}^{m_B} c^2 \; dm = c^2 [m]_{m_A}^{m_B}$$

(Here m_A and m_B are relativistic mass of the body at A with speed u_A and at B with speed u_B)

$\therefore$ Change in KE $(\Delta K) = (KE)_B - (KE)_A = (m_B - m_A)c^2 = (\Delta m)c^2$

i.e. **the KE gained / lost by a moving body is equal to c^2 times the increased / decreased in mass of the body.**

If we suppose that the body starts from rest at A $(u_A = 0)$, its initial KE will obviously be zero, i.e. $(KE)_A = 0$ and correspondingly mass will be equal to its rest mass, i.e. $m_A = m_0$. Thus the above relation becomes

$$(KE)_B = (m_B - m_0)c^2 \, ,$$

Which can be put into a general form by dropping the suffix B from the above expression as

$$KE = (m - m_0)c^2$$

$$\Rightarrow mc^2 = KE + m_0c^2$$

Here Einstein interpreted mc^2 as the **total energy** E. If the body is at rest, KE vanishes and the body is still possessing m_0c^2 amount of energy. This energy is defined as **internal or intrinsic or rest energy** of the body. The rest energy m_0c^2 includes all the possible type of energies (e.g. intermolecular potential energy, molecular translational energy, molecular vibrational energy, molecular rotational energy (thermal energy), electrical energy, etc.). Thus,

Total Energy = KE + Rest Energy

Or, $$E = mc^2 = KE + m_0c^2$$

Or, $$E = mc^2$$

$$\Rightarrow \text{Total Energy} = (\text{Relativistic mass}) \times c^2,$$

which is known as **mass-energy equivalence principle**. **It states that mass and energy are not two independent entities, they are different aspects of the same thing. According to the principle mass can be created or destroyed, but when this happens, an equivalent amount of energy simultaneously vanishes or comes into being.**

- The principle states the universal equivalence of mass and energy. The mass and energy of the universe are not conserved separately, but they are conserved as a whole. In classical mechanics conservation of mass and energy are treated as two basic principles and it is supposed that they are satisfied in any process in the reign of classical physics.
- In our day to day life, the conversion of energy to mass and vice-versa are not observed frequently, because this conversion is again restricted by some other very fundamental conservation principles, e.g. conservation of lepton number, conservation of baryon (proton +neutron) number, etc. According to the mass-energy equivalence principle, if 1mg of sand is converted to energy, we have to have 9×10^{10} J of energy. But the conversion of the whole mass into energy is not permitted by the above said conservation principles, because if the whole amount of sand gets converted to energy, the baryon number (total number of proton + neutron) will not conserved.
- We could **not** use the Newtonian expression for KE of a body moving with high speed comparable to the speed c as

$$KE = \frac{1}{2}mu^2 ,$$

where m is the relativistic mass of the body moving with speed u $(\approx c)$. For understanding the meaning, we have to do the following simple mathematics.

$$KE = E - m_0c^2 = (m - m_0)c^2$$

$$= \left[1 - \left(1 - \frac{u^2}{c^2}\right)^{1/2}\right]mc^2$$

$$= \left[1 - \left(1 - \frac{1}{2}\frac{u^2}{c^2} - \frac{1}{8}\left(\frac{u^2}{c^2}\right)^2 - \frac{3}{48}\left(\frac{u^2}{c^2}\right)^3 - \ldots\ldots\ldots\ldots\right)\right]mc^2$$

$$= \left[1 + \frac{1}{4}\left(\frac{u}{c}\right)^2 + \frac{3}{48}\left(\frac{u}{c}\right)^4 + \ldots\ldots\ldots\ldots\right]\frac{1}{2}mu^2$$

In relativistic mechanics, $u \approx c$ and so we can not neglect $\frac{u}{c}$ and its higher order terms in the above equation. Hence, in relativistic mechanics

$$KE \neq \frac{1}{2}mu^2 .$$

- Now we repeat the same calculation in classical limit when $\frac{u}{c} << 1$, the speed of the body is very much smaller than c, e.g. a cricket ball moving with speed of 100 km/h ≈ 28 m/s. The speed of light in free space is 3×10^8 m/s and so $\frac{u}{c} = 0.0000000933$.

$$KE = E - m_0c^2 = (m - m_0)c^2$$

$$= \left[\left(1 - \frac{u^2}{c^2}\right)^{-1/2} - 1\right]m_0c^2$$

$$=\left[\left(\begin{array}{l}1+\frac{1}{2}\frac{u^2}{c^2}+\frac{3}{8}\left(\frac{u^2}{c^2}\right)^2 \\ \qquad +\frac{15}{48}\left(\frac{u^2}{c^2}\right)^3+\ldots\ldots\ldots\ldots\end{array}\right)-1\right]m_0c^2$$

$$=\left[1+\frac{3}{4}\left(\frac{u^2}{c^2}\right)+\frac{15}{24}\left(\frac{u^2}{c^2}\right)^2+\ldots\ldots\ldots\ldots\right]\frac{1}{2}m_0u^2$$

Hence in classical limit (i.e. in Newtonian Mechanics) $KE=\frac{1}{2}m_0u^2$

- **Concept of Relativistic Mass**

In classical physics, the mass of the bodies are assumed to be constants and as a consequence, in a two body collision problem under action–reaction forces, the quantity $m_1\vec{u}_1+m_2\vec{u}_2$ (total linear momentum) remains the same before and after collision, which is the law of conservation of linear momentum of classical mechanics. If the classical concept of mass is applied in Special Relativity, it is seen that the above mention quantity may increase or decrease after a collision. But the Lorentz-Einstein transformation shows that there is a corresponding quantity $\frac{m_1}{\sqrt{1-\frac{u_1^2}{c^2}}}\vec{u}_1+\frac{m_2}{\sqrt{1-\frac{u_2^2}{c^2}}}\vec{u}_2$ remains conserved. If the quantity $\frac{m}{\sqrt{1-\frac{u^2}{c^2}}}$ is defined as mass, then the total linear momentum of the process would again be conserved. This mass is more correctly called as relativistic mass of the body moving with speed u. At rest, when u=0, this mass is said to be rest mass and it is then denoted by m_0.

- **Concept of Rest Mass Energy and Rest Mass of a body**

KE of a body due to its 'as a whole motion' (translational + rotational + vibrational) may be regarded as external energy and by subtracting it from the total energy $E=mc^2$ we can get the rest mass energy m_0c^2 or total internal energy which includes energies due to all molecular motions (thermal energy), intermolecular potential energies, atomic potential energy, nuclear potential energy, etc. So,

$m_0 = \frac{1}{c^2} \times$ **(**KE due to molecular motions + intermolecular PE + atomic PE + nuclear PE +……………**)**

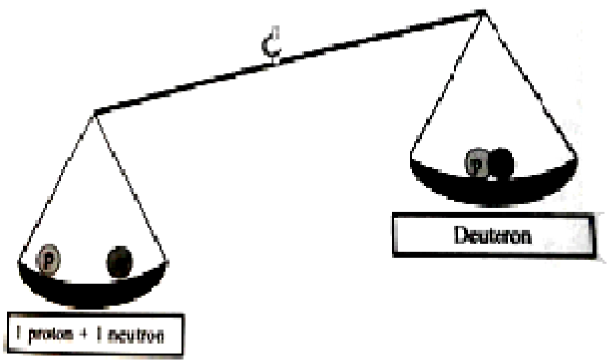
Internal energy of a body does not remain constant for ever, because during heat transfer, molecular KE changes; chemical reaction changes intermolecular PE and atomic PE, nuclear reaction changes nuclear PE, etc. Therefore, rest mass of a body never be a constant. Greater the internal energy, greater is the rest mass. In relativistic picture, the rest mass reflects the internal energy of an object. So, **a potato becomes heavier when it is heated up.** Similarly a compressed spring with additional PE is heavier than a released spring.

Application of mass-energy equivalence principle

1) In the formation of a nucleus, the nucleons (proton + neutron) have lost some amount of their mass (**mass defect)** and that lost mass gets converted to energy, called **binding energy** (BE), which is required to bound all the nucleons together in a small space. This BE is equal to c2 times the lost mass.

$$BE = c^2 \times \left[\left(n_p m_p + n_n m_n \right) - m_{nucl} \right]$$

Where n_p and n_n are respectively the number of proton and neutron and m_p, m_n and m_{nucl} are respectively the mass of proton, neutron and nucleus.



2) In case of ($e^- - e^+$) **pair production** in cosmic ray shower, energy is found to be converted to mass and in ($e^- - e^+$) **pair annihilation** process, mass gets converted to energy.

3) In nuclear fission reaction, the heavy nucleus like uranium may form fission fragments. The total rest mass of all the fragments is less than that of the original

heavy nucleus. The decrease in mass Δm during the fission appears in the form of energy equal to $(\Delta m)c^2$ as given by the mass-energy equivalence principle.

$$U_{92}^{235} + n_0^1 \rightarrow Ba_{56}^{141} + Kr_{36}^{92} + 3n_0^1 + Energy$$

The liberation of tremendous amount of energy due to conversion of mass into energy in an uncontrolled chain reaction is the **basic principle of atom bomb**. And in **nuclear reactor**, it is allowed to initiate in a controlled manner.

4) When two light nuclei like hydrogen or its isotopes combine to form a heavy nucleus undergoing the process of fusion, a tremendous amount of energy is released, which is the **basic principle of hydrogen bomb.**

$$H_1^2 + H_1^2 \rightarrow He_2^3 + n + Energy$$

The degree of temperature and pressure to carry out the process is really high. **Why?** The source of heat and light radiation of a star is solely the nuclear fusion reaction which is occurring at its core.

- Einstein's law displaced the old law of the conservation of mass, worked out by Lavoisier, which says that matter, understood as mass, can neither be created nor destroyed. In fact, every chemical reaction that releases energy converts a small amount of mass into energy. This could not be measured in the kind of chemical reaction known to the 19th century, such as the burning of coal. But nuclear reaction releases sufficient energy to reveal a measurable loss of mass. All matter, even when at "rest," contains staggering amounts of energy. However, as this cannot be observed, it was not understood until Einstein explained it.

Relativistic momentum

The relativistic mass of a body multiplied it by its velocity is defined as relativistic momentum in Special Relativity, i.e.

$$\vec{p} = m\vec{u} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\vec{u}\ .$$

Relation between relativistic momentum and KE

Using the mass variation formula in the expression $mc^2 = KE + m_0c^2$ and then squaring both sides and simplifying and keeping the terms involving KE (K) in one side and then applying the concept of relativistic momentum, we have

$$p^2c^2 = k + 2km_0c^2,$$

whereas in classical physics, it is in the form

$$K = \frac{1}{2}m_0u^2.$$

Relation between relativistic momentum and total energy

Squaring both sides of the expression $E = mc^2 = KE + m_0c^2$ and then applying the equation $p^2c^2 = k + 2km_0c^2$, we have got the following expression

$$E^2 = p^2c^2 + m_0^2c^4$$

Lorentz invariant

If p and E are the relativistic momentum and energy of a body of rest mass m_0 in S inertial frame and p' and E' are the corresponding values with respect to another inertial frame S' moving with certain uniform velocity w. r. to S frame, then by the above equation

$$E^2 - p^2c^2 = E'^2 - p'^2c^2 = m_0^2c^4$$

Since $m_0^2c^4$ is a constant quantity independent of the frame of reference, the quantity $\left(E^2 - p^2c^2\right)$ must be an invariant under Lorentz-Einstein transformation, that is to say that the quantity remains invariant in all inertial frames S, S', S'', etc. Such types of quantities, which remain unchanged under Lorentz-Einstein transformation, are called Lorentz invariant. Conceptually they are similar to **Galilean invariants** like space interval, time interval, acceleration, force etc., which remain invariant under Galilean transformation. So, invariants are always subject to particular transformations.

Zero Rest Mass

The relation between total energy *E* and momentum *p* in the relativistic mechanics is

$$E^2 = p^2c^2 + m_0^2c^4 .$$

From the mass-energy equivalence principle $E = mc^2$ and momentum and velocity relation $\vec{p} = m\vec{u}$, we have

$$p = \frac{mc^2}{c^2}u \Rightarrow pc = E\frac{u}{c} ,$$

For light waves or photons moving with the speed c, the above relation becomes

$$E = pc .$$

Both the above relations for total energy of a photon will be mathematically consistent only when the rest mass of photon vanishes, i.e. $m_0 = 0$. So, we conclude that **any particle moving with speed equal to c must have zero rest mass or in the other way we can state as all the particles with zero rest mass propagate with the speed of light c in free space.**

The above mathematics also reveals that material particles with finite rest mass, how much small it may be, always move with speed less than that of light in free space.

For a material particle, $m_0 \neq 0$

So, $E^2 = p^2c^2 + m_0^2c^4 \Rightarrow E > pc$

Again, $\vec{p} = m\vec{u}$ $p = \frac{mc^2}{c^2}u$ and hence $pc = E\left(\frac{u}{c}\right)$

Since, $E > pc,$ so $u < c$

It therefore indicates indirectly that for material particle there is a **limiting speed equal to c**.

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Michelson-Morley Experiment

Ether

Newton (1643-1727) proposed the **corpuscular theory of light**, while Huygens (1629-1695) put forward the **wave theory of light**. Initially Newton's theory was welcomed by all, but the work of Young and Fresnel on interference and diffraction of light showed clearly the validity of wave theory, completely rejecting the Newton's theory. The concept of light as a wave process in a medium thus established, and the theory of light was reduced to the theory of oscillations in a **medium** that fills the entire universe.

Ether Hypothesis

The medium, that pervades throughout the whole universe and helps the light to propagate with the speed equal to c through it, was hypothesized as the ether.

Absolute velocity

The ether medium or ether frame remains **stationary** in space and the **motion** of a body relative to the ether was supposed to be the absolute, and relative w. r. to other moving frames. The velocity of a body w. r. to ether was called the **absolute velocity** of the body and it was supposed to be independent of the motion of other bodies.

If we know the absolute velocity $(\vec{V}_A)$ of a body A w. r. to ether, and if the relative velocity $(\vec{V}_{BA})$ of another body B w. r. to the body A can be measured by doing an experiment, then the absolute velocity $(\vec{V}_B)$ of the later one could be found out (since $\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$). That is the trick brilliantly applied by Michelson and Morley in his famous optical experiment.

Galilean Relativity, Maxwell's theory of electromagnetism and findings of the M-M Experiment

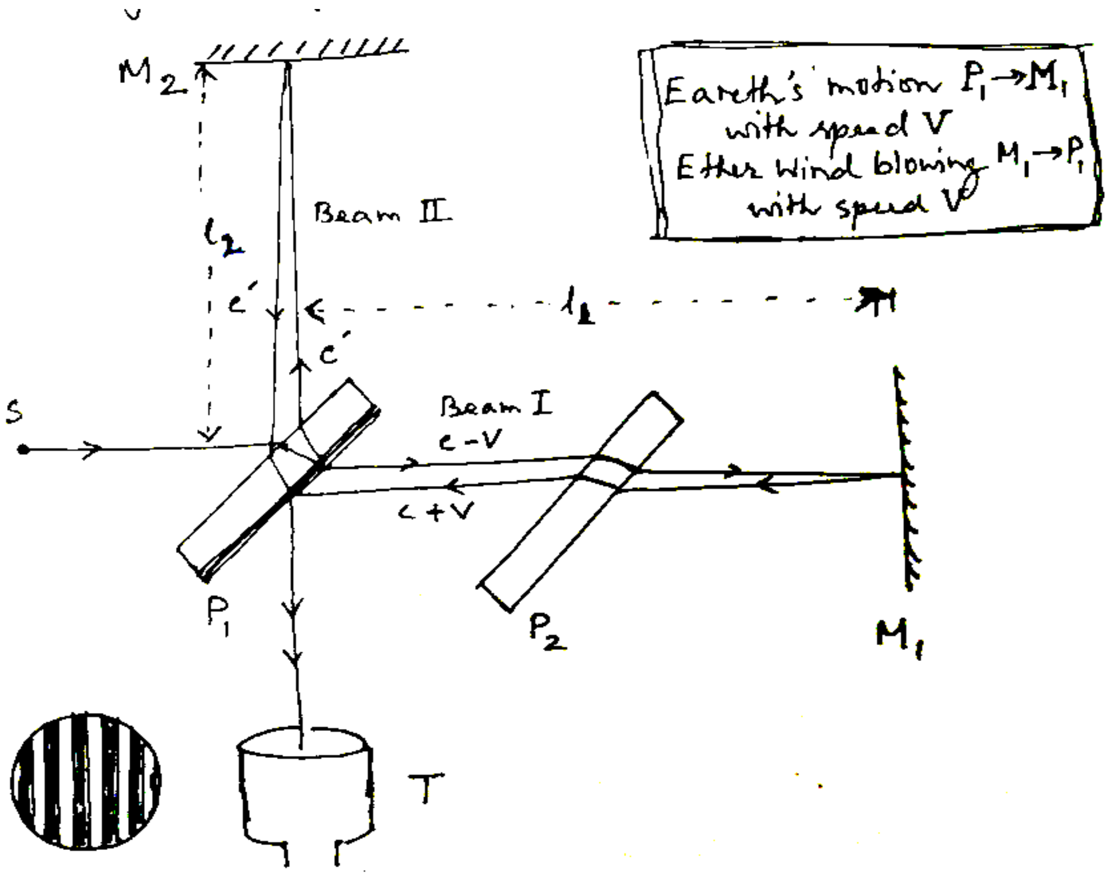
If we accept both Galilean Relativity and Maxwell's theory of electromagnetism as basically correct, Maxwell's wave equation for propagation of light only holds in a unique privileged frame where the speed of light in vacuum is equal to c. This absolute frame was hypothesized as the ether frame, the ether medium that pervades the whole universe and it is also essential for the propagation of light.

For detection of that absolute frame, i.e. the ether medium, Physicists performed **optical experiments**. Some of the famous experiments are Astronomical Aberration Experiment, Fizeau's Experiment, Michelson-Morley (M-M) Experiment, etc.

Out of all other experiments, M-M experiment (1881 - 87) was able to play a decisive role. It was the acid test for detection of ether medium. The American Physicist A. A. Michelson, later aided by E. W. Morley carried out a series of experiments to measure the speed of earth w. r. to ether assuming that the speed of light is equal to c w. r. to ether as given by Maxwell's theory and there is a **ether wind** blowing past the earth as the earth is moving through ether (just like the air wind blowing past a motorcyclist pushing his hair in backward direction) and this ether wind would alter the speed of light in the similar way the air wind effects on the speed of sound. They thought that they could detect the change of speed of light on earth due to the effect of ether wind and from this, they could measure the speed of earth w. r. to ether. If they could, then it would establish the existence of ether. But they got a **null result (negative result),** they could not measure the speed of earth through ether and therefore they were unable to establish that there is ether. It was totally an unexpected result and Michelson thought that somehow the ether wind disappeared during the experiment and so they got a –ve result. So, they performed the same experiment at an interval of six months when the direction of motion of the earth in its orbit became just opposite to that six months back. But their experiment again yielded a null result. Michelson and Morley improved the resolving power of their apparatus by modifying its design and performed the experiment at different altitudes, at different seasons in a year expecting the effect of ether wind on light propagation, but every time they got null result. They were unable to measure the speed of earth through ether. The conclusion drawn from their experimental result is that since the speed of earth could not be measured w. r. to ether, the existence of ether cannot be confirmed and so the ether concept could be discarded, there is no meaning of hypothesing an absolute ether frame where the light moves with the constant speed equal to c. And it is established beyond doubt that the speed of light through free space is simply a constant equal to c irrespective of the motion of source and observer. (The situation is like the measuring the speed of a moving boat w. r. to the still water in a lake by performing an experiment on the boat, assuming that the experimentalist cannot see the water or cannot feel the presence of water in the lake through any other means. If the person on the boat cannot measure the speed of the boat w. r. to the still water, he readily comes to the conclusion that there is no water in the lake or if there is water, the existence of water is meaningless to him.!!!)

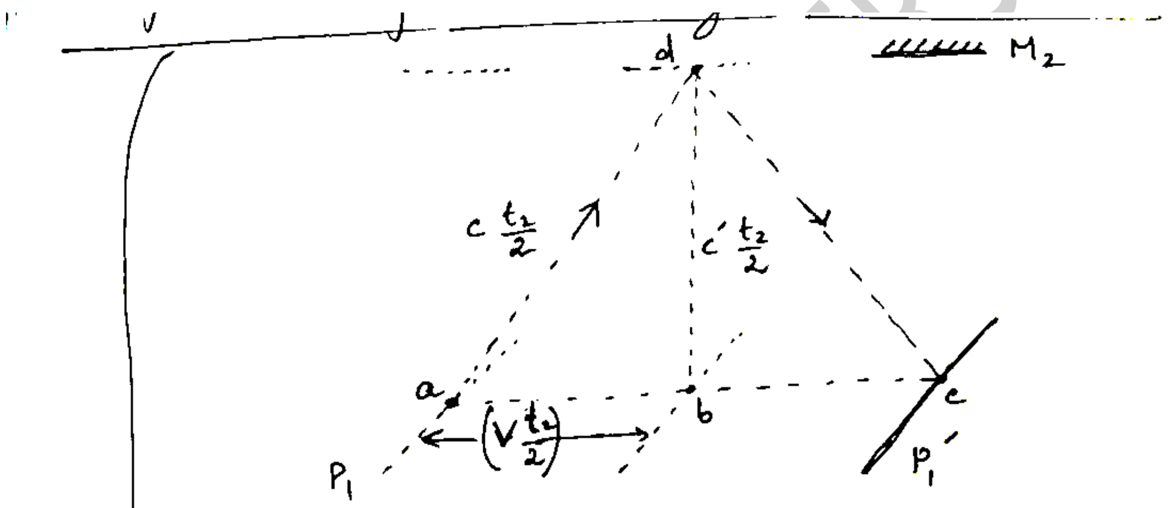
Aim of the experiment:

Michelson and Morley carried out a series of experiments just to measure the speed of earth through the ether medium assuming that there was a 'ether wind' blowing past the earth, as the earth is moving through the ether and this 'ether wind' would alter the speed of light in the similar way the air wind effects on the speed of sound.



Michelson and Morley used an interferometer of remarkable sensitivity invented by them. In the experiment, the light from the source S is split into two mutually perpendicular beams by a **half-silvered** plate (on the back) P_1. These two beams are made to reflect from two mirrors M_1 and M_2, which are placed normal to their paths at almost equal distances from the plate P_1. The mirrors M_1 and M_2 are optically flat and **heavily silvered** on the front face to avoid multiple reflections (to minimise the amount of absorbed energy on the glass plate) and are arranged right angled to each other. To make the **optical paths for the two beams I and II equal** through glass, a compensating plate P_2 identical with the plate P_1 but not silvered is arranged parallel to plate P_1 in the path of beam I. The two beams I and II, after suffering reflections on the mirrors M_1 and M_2 respectively are allowed to reunite and the reunited beam is observed through a telescope T and an interference pattern on the field of view of the telescope is observed. The time difference between the two beams I and II that arises due to their up and down journey across the distances l_1 and l_2 respectively is the very cause of formation of interference pattern on the field of view of the telescope.

Let us suppose that the earth along with the whole experimental set up is moving with speed V w.r. to the ether medium in the direction P_1 to M_1 and so 'ether wind' blows with speed V in the direction from M_1 to P_1. The ether wind effects the propagation of light beam I and so the speed of light beam I in its up journey becomes $(c\text{-}V)$ and in its down journey, it is $(c+V)$. (imagine the effect of wind on sound wave). For the beam II, the speed of light becomes $c^{/} = \sqrt{c^2 - V^2}$ in the perpendicular directions $P_1 \rightleftarrows M_1$ w.r. to the direction of motion of earth through the ether.



Additional Note

Time required by the beam II in its up-down journey is t_2 and so the distance travelled by the plate P_1 through space is $V t_2$. If c' be the speed of light beam II in the direction perpendicular to the direction of motion of earth, using the right-angled triangle abd

$$(ad)^2 = (ab)^2 + (bd)^2$$

$$\Rightarrow \left(c\,\frac{t_2}{2}\right)^2 = \left(V\,\frac{t_2}{2}\right)^2 + \left(c'\,\frac{t_2}{2}\right)^2 \Rightarrow c'^2 = c^2 - V^2$$

If t_1 and t_2 be the time intervals taken by the two beams I and II respectively for their up and down journey between the plate P_1 and respective mirror M_1/M_2, then

$$t_1 = \frac{l_1}{c-V} + \frac{l_1}{c+V} = \frac{2l_1}{c}\left(\frac{1}{1-\frac{V^2}{c^2}}\right)$$

$$t_2 = 2\frac{l_2}{\sqrt{c^2-V^2}} = \frac{2l_2}{c}\left(\frac{1}{1-\frac{V^2}{c^2}}\right)^{1/2}$$

And hence the difference in time at the time of reuniting of the two beams at the backside of the plate P_1 is

$$\Delta t = t_1 - t_2 = \frac{2l_1}{c}\left(\frac{1}{1-\frac{V^2}{c^2}}\right) - \frac{2l_2}{c}\left(\frac{1}{1-\frac{V^2}{c^2}}\right)^{1/2}$$

If we suppose V=0 i.e. no ether wind, then also $\Delta t \neq 0$ and still the interference fringe pattern would be observed through the telescope, since this Δt is the actual factor that governs the fringe pattern. So, observation of interference pattern does not confirm the effect of ether wind on the propagation of light wave and so, the existence of ether.

To observe the effect of ether wind, the whole assemblage is given a 90^0 rotation in its own plane and the roles of the two beams are thereby got reversed. For this resulting situation, $t_1^{/}$ and $t_2^{/}$ are the time intervals required by the beam I and II respectively for their up and down journey. So

$$t_1^{/} = \frac{2l_1}{c}\left(\frac{1}{1-\frac{V^2}{c^2}}\right)^{1/2}$$

$$t_2^{/} = \frac{2l_2}{c}\left(\frac{1}{1-\frac{V^2}{c^2}}\right)$$

And the difference in time

$$\Delta t^{/} = t_1^{/} - t_2^{/} = \frac{2l_1}{c}\left(\frac{1}{1-\frac{V^2}{c^2}}\right)^{1/2} - \frac{2l_2}{c}\left(\frac{1}{1-\frac{V^2}{c^2}}\right)$$

Since $\Delta t^{/} \neq 0$, an interference pattern is again observed through the telescope after giving 90^0 rotation to the whole assemblage.

Since $\Delta t \neq \Delta t^{/}$, the fringe system would appear to be shifted laterally after giving 90^0 rotation. Theoretically, the number of fringes shifted in the fringe system is given by

$$\delta n = \left(\frac{l_1 + l_2}{\lambda}\right)\frac{V^2}{c^2}$$

But such shift to the fringe system is not bserved.

$$\delta n = \delta(\Delta t)\,\frac{c}{\lambda}$$

$$= (\Delta t' \sim \Delta t)\,\frac{c}{\lambda}$$

$$= \frac{2(l_1+l_2)}{c}\left[\left(\frac{1}{1-v^2/c^2}\right)^{1/2} \sim \left(\frac{1}{1-v^2/c^2}\right)\right]\frac{c}{\lambda}$$

$$= \frac{2(l_1+l_2)}{\lambda}\left[\left(1-v^2/c^2\right)^{-1/2} \sim \left(1-v^2/c^2\right)^{-1}\right]$$

$$= \frac{2(l_1+l_2)}{\lambda}\left[\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) \sim \left(1+\frac{v^2}{c^2}\right)\right]$$

$$\left(\because \frac{v}{c} \ll 1\right)$$

$$\text{or} \quad \boxed{\delta n = \left(\frac{l_1+l_2}{\lambda}\right)\frac{v^2}{c^2}}$$

Result of Michelson and Morley Experiment:

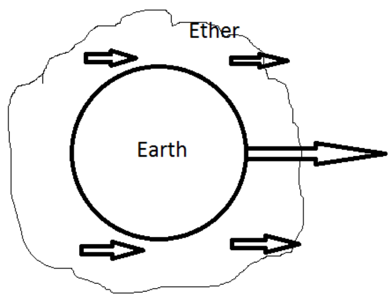
If a shift of fringe pattern is observed, the speed of ether wind and so the speed of earth w.r. to ether (i.e. V) can be calculated by measuring the number of fringe got shifted (δn) after giving 90^0 rotation. But Michelson and Morley observed no such shift of fringe patter after giving 90^0 rotation of the whole experimental set-up, and so they were unable to calculate the speed of the ether wind and so the speed of earth w.r. to ether (i.e. V), that is to say **they got a null result (or negative result).** They improved the resolving power of the apparatus by modifying its design and also perform the same experiment at different altitudes at different seasons in a year expecting the effect of ether wind on the speed of light, but every time they ended up with a null result.

Conclusion:

Michelson and Morley could not observe any lateral shifting of the fringe pattern after giving 90^0 rotation of the whole experimental set-up. It happens only when speed of light in free space remains constant irrespective of the motion of the observer as well as of the source. It indirectly means that no ether medium is required, where only the speed of light was assumed to be equal to c. And so, the ether concept is thereafter discarded.

Some fragile attempts made to restore the status of ether

The conclusion drawn from M-M experiment was not welcomed by the Physics community at that time for the general reason that the light needs a medium for propagation and ether provides that medium. Michelson himself was not happy with the conclusion and tried to explain the –ve result of his experiment by proposing a hypothesis, called '**ether drag hypothesis'.** According to the said hypothesis, Michelson assumed that the whole volume of ether in the surroundings of earth was constantly dragged by the earth as the earth is moving through the ether medium, if it were so, there would not be any ether wind on the surface of earth and there were no question of alternation of the speed of light by ether wind. Because of that, Michelson and Morley got null result.



But the hypothesis was unable to support the M-M experimental result. Because, the hypothesis has an inherent defect as it goes against the idea of a calm ether sea at **absolute** rest. Again the hypothesis could not be established by some other experiments (e.g. Bradley's Astronomical Aberration Experiment, Fizeau's experiment, etc.).

Though the physicists had to keep aside the ether drag theory, still they were not brave enough to accept the null result and constantly they were trying to formulate new theories to defend the –ve result and thereby try to establish the existence of ether. One of the attempts was made by FitzGerald. He proposed the '**length contraction hypothesis'**, according to which all material bodies moving w .r. to stationary ether with speed V got contracted by the factor $\sqrt{1-\frac{V^2}{c^2}}$ in the direction of motion, while the dimension perpendicular to the direction of motion remained unaltered. If we apply this contraction to the M-M experiment, then the –ve result can be explained beautifully. But the experiment performed by Kennedy and Thorndike showed the null effect of FitzGerald contraction. So, -ve result remained unexplainable in presence of ether.

The final blow on ether

At the first glance, all the hypotheses proposed for the sake of ether are found to explain the null result of M-M experiment in presence of ether, but they are all **ad hoc in nature** and so they are far from convincing. From the work of **Poincare/** and **Einstein** in the development of relativity theory, it was revealed that there is absolutely no place for those hypothesis and the physicists ultimately have to accept that no absolute frame like ether is required to hold the Maxwell's equations for electromagnetism and speed of light is a constant equal to c w. r. to any frame.

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Minkowski's four dimensional continuum

In classical physics space and time are independwnt and so when space coordinates x, y and z of one inertial frame is transformed to another, the time coordinate t remains unaffected. In relativity, however, space and time are not independent. The time coordinate of one inertial frame depends on both the space and time coordinates of another inertial frame. By Lorentz-Einstein transformation equation

$$t' = \frac{t - \frac{V}{c^2}x}{\sqrt{1-\frac{V^2}{c^2}}},$$

it is found that space and time get entangled in relativity and H. Minkowski was the first to suggest that it is judicious to treat both of them together and he also clearly showed how this could be done.

It is convenient to express the results of Special Relativity by regarding events as occurring in a four-dimensional continuum, called **Minkowski space** or **spacetime continuum**. It is briefly referred as '**four-space**' or '**world**'. The coordinates chosen (x, y, z & t) form an orthogonal coordinate system in for dimensions (3-space + 1-time). A point representing an event in Minkowski space is called a '**world point**'. As a particle moves in real space with time, its successive world points trace out a curve in that world, called '**world line'.**

Physical laws on the interaction of particles can be thought of as the geometric relation between their world lines. In this sense, Minkowski may be said to have geometrized physics.

For geometrical representation, we consider only one space axis x and the time axis t perpendicular to x-axis. Such a simplification does not lose any generality. For convenience, x-axis is taken as a horizontal one and t-axis is vertical. It is convenient to keep the dimensions of the coordinates x and t the same and for that time t is multiplied by the universal constant c (speed of light in vacuum). By putting ct=w, the Lorentz-Einstein transformation equations become-

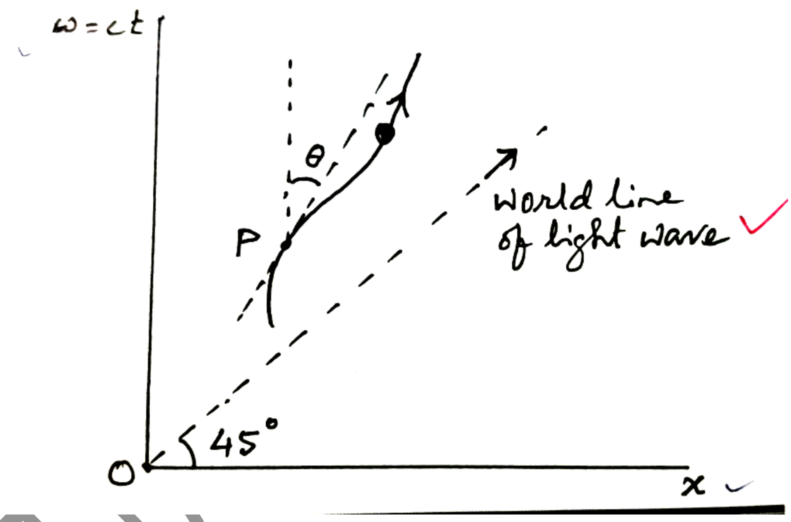
$$x' = \frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \Rightarrow x' = \alpha\left(x - \beta\omega\right)$$

and

$$t' = \frac{t - \frac{V}{c^2}x}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \Rightarrow \omega' = \alpha(\omega - \beta x)\,,$$

where $\alpha = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$ and $\beta = \frac{V}{c}$. There is a symmetry in the form of the above equations in the sense that we can go from one equation to the other by simply replacing one coordinate with the other ($x \leftrightarrow \omega$).

Diagrammatic representation of inertial frames under Minkowski's notation:



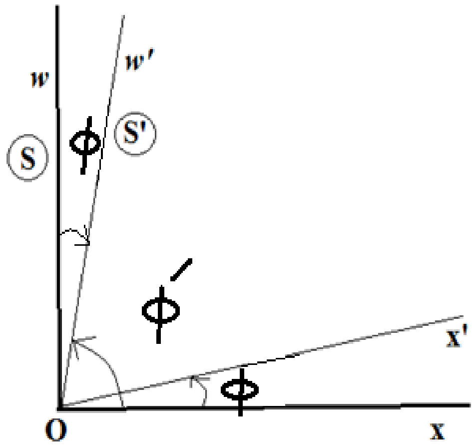
To represent the S inertial frame geometrically, we will draw the x and w axes perpendicularly to each other and the motion of a particle is represented by a world line. The tangent to the world line at a point P makes an angle θ with the w-axis. So the slope of the tangent with w-axis at P is given by

$$\tan\theta = \frac{dx}{d\omega} = \frac{1}{c}\frac{dx}{dt} = \frac{u}{c},$$

where u is the speed of the particle at P. Since $\frac{u}{c} < 1$ for all material particle, $\tan\theta < 1$ and hence $\theta < 45^0$. It means the tangent to the world line at any point is inclined at an angle less than 45^0 with the w-axis (greater than 45^0 with the x-axis).

For light wave, u=c and so $\tan\theta = 1 \Rightarrow \theta = 45^0$. It means that the world line of light wave is a straight line making an angle 45^0 with either of the axis.

Diagrammatic representation of a moving inertial frame w.r. to another inertial frame under Minkowski's notation:



To represent the inertial frame S$^{/}$ moving with respect to S frame with uniform speed V along +x-direction, we have to take the help of equations

$$x' = \alpha(x - \beta\omega) \quad \text{and}$$

$$\omega' = \alpha(\omega - \beta x).$$

To draw the x$^{/}$-axis for S$^{/}$ frame, we apply the following trick. Since $\omega' = 0$ along x$^{/}$ axis, the second equation yields $\omega = \beta x$, which means that x$^{/}$-axis is a straight line passing through the origin of ω-x diagram and makes and angle φ with x-axis such that its slope $\tan\varphi = \beta$.

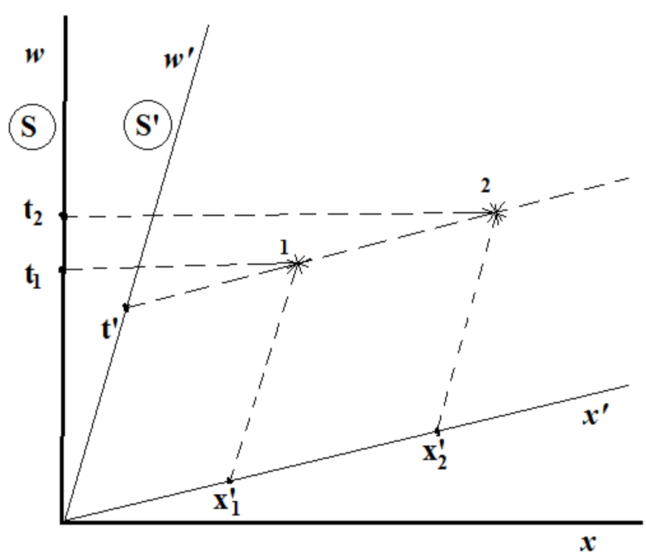
Again along ω'-axis in S$^{/}$ frame, $x' = 0$ and hence from the first equation, we get $\omega = \frac{1}{\beta}x$, which means that ω'-axis is a straight line passing through the origin of ω-x diagram making an angle φ' with x-axis such that its slope $\tan\varphi' = \frac{1}{\beta}$.

Since $\tan\varphi = \beta$ and $\tan\varphi' = \frac{1}{\beta}$, $\varphi' = \frac{\pi}{2} - \varphi$, and it indicates that the angle made by x$^{/}$-axis with x-axis is exactly equal to the angle made by ω$^{/}$-axis with ω-axis.

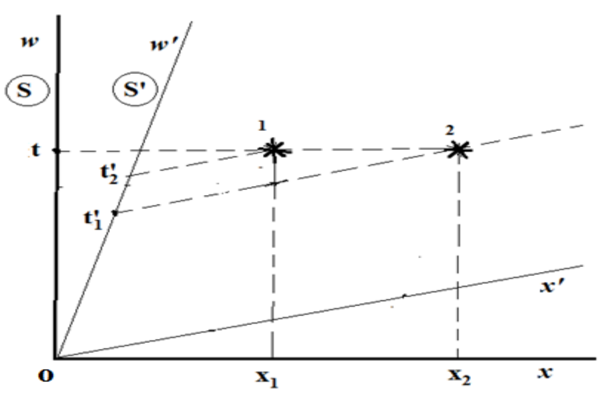
The figure shows that $\omega^/$- $x^/$ system is a non-orthogonal coordinate system. Since $x^/$ and $\omega^/$ axes are obtained by using Lorentz-Einstein transformation equations, it can be said that the Lorentz-Einstein transformation transforms an orthogonal system to a non-orthogonal one.

Use of Minkowski's diagram

a) Simultaneity is relative? Explain



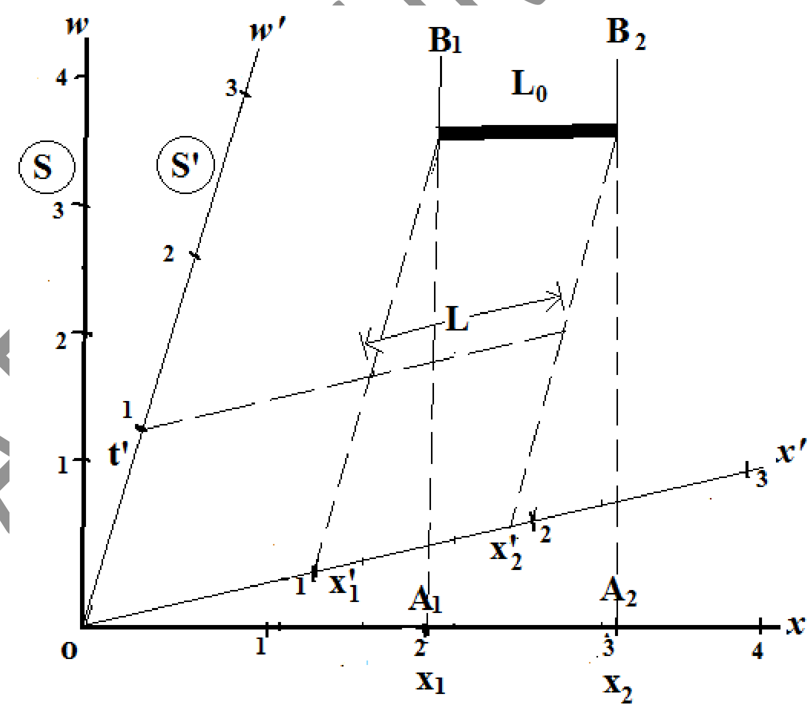
We suppose that two events 1 and 2 occurred simultaneously at the instant $t^/$ ($=t_1^/= t_2^/$) in the moving frame $S^/$ at two points $x_1^/$ and $x_2^/$. But those two events are appeared to occur at t_1 and t_2 from the S frame and $t_1< t_2$, i.e. event 1 occurred first, after than event 2 occurred. It shows that the events which are simultaneous in $S^/$ frame are found to have time sequence in another frame S. It means that simultaneity is not an absolute concept. Two simultaneous events for one frame never be simultaneous for another frame.



The same conclusion can be drawn by considering two simultaneous events 1 & 2 occurred in S frame at the instant t at x_1 and x_2. They are found to occur at $t_1^/$ and $t_2^/$ instants of time, there is a time order in their occurrence in $S^/$ frame, though they are simultaneous events at S frame.

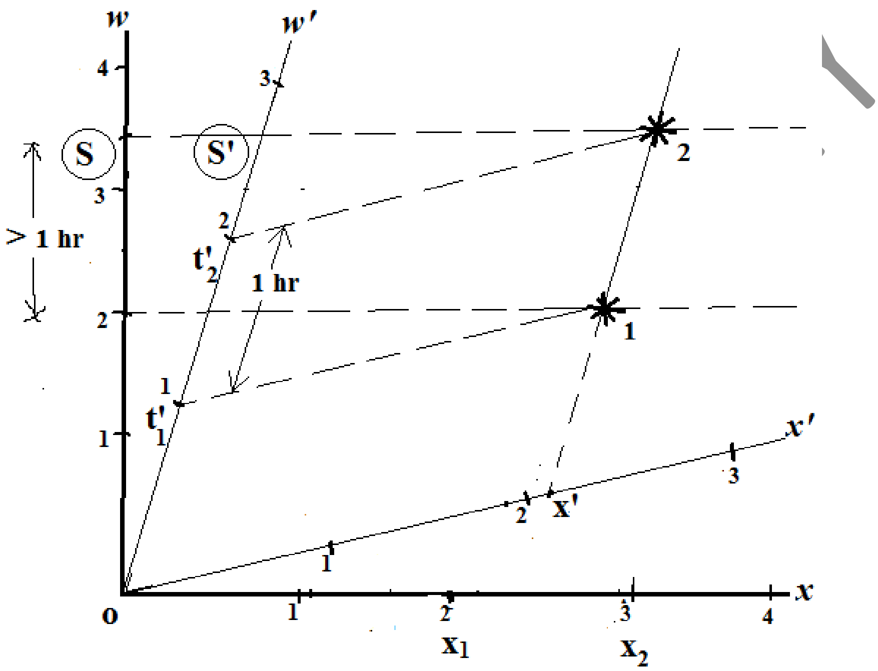
b) Length contraction

We suppose that a rod of length 1m ($L_0=A_1A_2=x_2-x_1$) is at rest in S inertial frame. A_1B_1 and A_2B_2 are the world lines for the two ends 1 and 2 respectively in ω-x spacetime corresponding to S frame. For an observer in $S^/$ frame moving with uniform velocity with respect to S frame (along X-axis, say), the rod at rest in S frame is a moving one and so if he wants to measure the length of the rod placed at rest in S frame, he must have to take the readings $x_1^/$ & $x_2^/$ of its two ends 1 & 2 simultaneously at $t^/$ (say) instant of time on $x^/$-axis. It is seen clearly from the Minkowski's diagram that the length of the moving rod as observed from $S^/$ frame $L^/= x_2^/- x_1^/$ is smaller than L (=1m) for the stationary rod in S frame, i.e. the moving rod gets contracted.



a) Time dilation

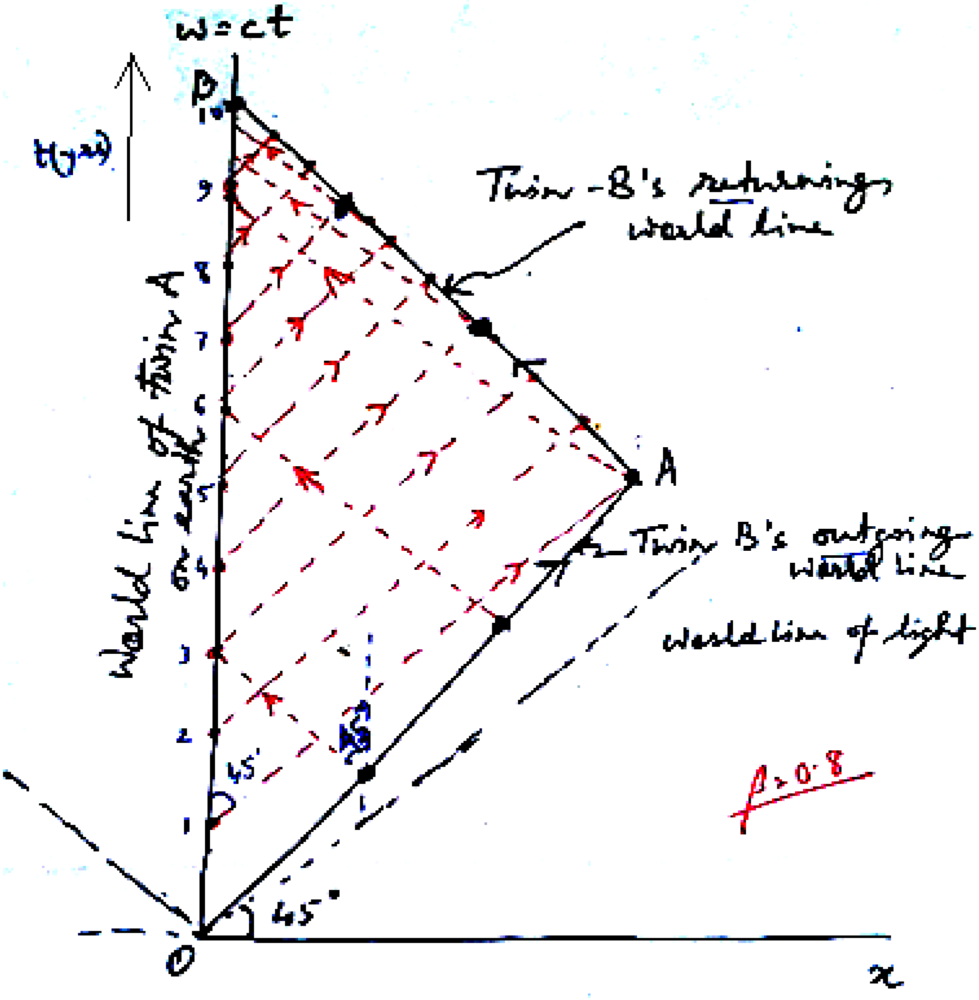
We suppose that two events 1 & 2 occurred one after the other at the same point $x^/$ at the instant of time $t_1^/$ & $t_2^/$ in $S^/$ inertial frame moving with uniform speed along X-direction w.r. to S inertial frame and the time interval $\Delta t' = t_2' - t_1' = 1$hour (proper time interval). But the time of ocuurence of the events from S frame recorded by the clock from S frame are found to be t_1 and t_2 respectively for the events 1 and 2 and from the Minkowski's diagram, it is seen that the time interval $\Delta t = t_2 - t_1$ is found to be longer than 1 hour. It indicates that time goes fast in the rest frame S, while it goes slow in the moving frame $S^{/}$, that is to say time gets retarded or dilated in a moving frame.



b) Twin paradox

Twin A is on Earth and twin B goes for a space voyage riding on a spaceship moving at a speed of 0.8c relative to twin A. World line of twin A is a straight line along w (=ct)- axis and that of for twin B in its outgoing journey is inclined to w-axis less than 45^0. When twin B sends a signal in every one year interval, the twin A receives it after every three years of interval. Thus, the time goes slow inside the spaceship of twin B with respect to twin A. So, Twin A confirms that twin B remains younger than him. It is shown by drawing the world line for each signal sent by twin B to twin A. When six (06) years completed for twin B as shown by the dots on twin B's world line, ten (10) years elapsed for twin A as shown by the dots on twin A's world line.

From the point of view of twin B, twin A is moving away from twin B with speed -0.8c. Twin A starts sending a signal to twin B in every one year of interval, but twin B receives them in every three years of interval. It is shown by the world lines for the signals sent by twin A to twin B.

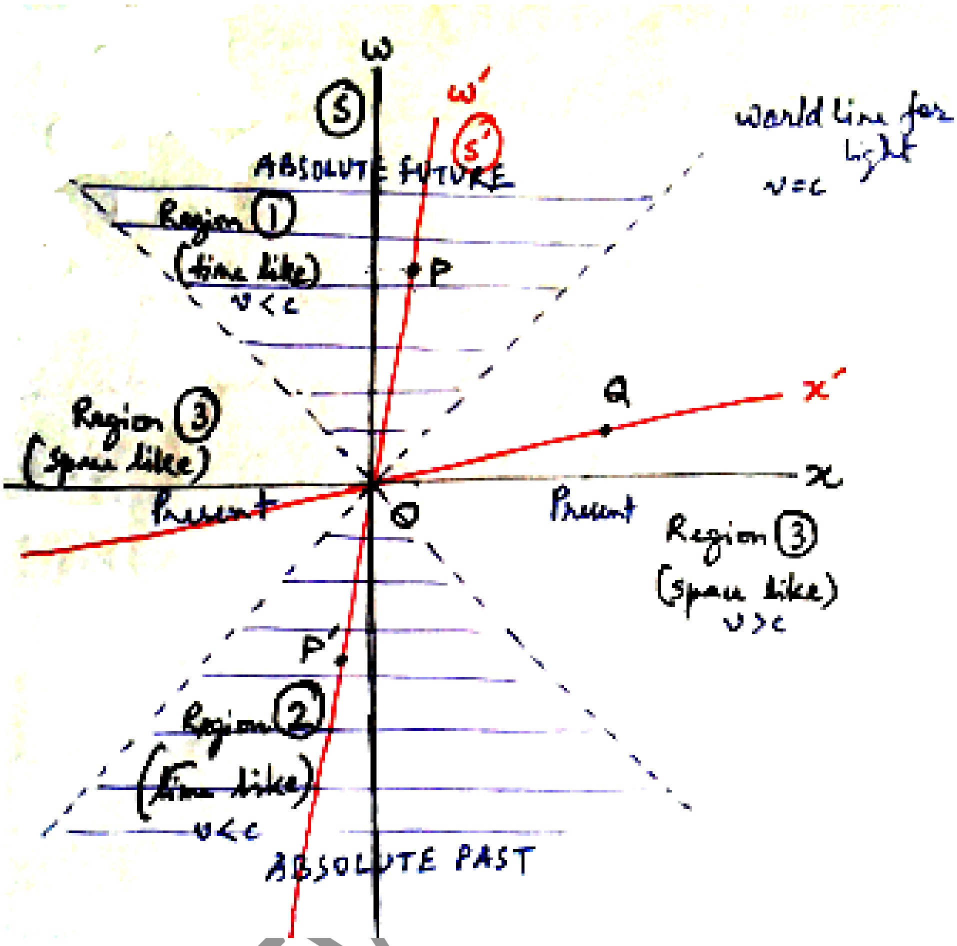


c) Past Present and future

Two inertial frames S and $S^/$ are represented by w-x and $w^/$- $x^/$ coordinate systems in Minkowski's diagram. $S^/$ frame is moving with constant velocity w.r. to S frame.

i) In region 1 bounded by the world lines of light waves,

the events at O and P in $S^/$ frame occur at the same place ($x^/$=0), but at different instant of time and the event P follows the event O and it is true for any event on the upper half of the shaded area (region 1). All the events in the region 1 are absolutely in the future relative to event O and so this region is called **Absolute Future**.



ii) In **region 2**, i.e. lower half of the shaded area,

for any event at $P^{/}$ preceded event O in time in $S^{/}$ frame, but occurred at the same place ($x^{/}=0$). S the events in the region 2 are absolutely in the past relative to the event O and therefore this region is called **Absolute Past**.

Thus in regions 1 and 2, for all events there is a time order relative to event O without any definite space order always and so these two regions are called **time-like** and the world interval OP or $OP^{/}$ is called **time-like interval.** For world line there, the velocity (u) of a body is always less than c and so two events can **communicate** (since signals are propagating with speed c).

i) **In region 3,** i.e. in the undashed region,

Both events O and Q on x' axis in S' frame occur at the same time ($w^{/}=0$), but they are separated only in space. Thus the events O and Q appear to be simultaneous in region 3 and so, this region 3 is called the **present**. The events O and Q are separated in space order rather than in time order and so the region 3 is said to be **space-like**. The speed of anything (particles as well as signals) for communication should be greater than c. Since c is the limiting speed for all, two events in region 3 never communicate and hence this region is called **Present**.

Spacetime interval

In an Euclidean space, the separation between two points is measured by the distance between the points. A distance is purely spatial and is always +ve. In spacetime, the separation between two events is measured by the interval between the two events which has taken into account not only the **spatial separation** between the events but also their **temporal separation**. The **spacetime interval** between two events is defined as

$$\Delta s^2 = \Delta r^2 - c^2 \Delta t^2$$

Where c is the speed of light in free space and Δr and Δt denote the differences of the space and time coordinates respectively between the events.

Spacetime intervals may be classified into three distinct types: i) Time-like interval ii) Space-like interval and iii) light-like or null interval.

i) Time-like interval

Here $c^2\Delta t^2 > \Delta r^2 \Rightarrow \Delta s^2 < 0$. If the two events are separated by time-like interval, enough time passes between them for there to be a cause-effect relationship between the events. For a particle travelling through space at less than the speed of light c, any two events which occur to or by the particle must be separated by a time-like interval. Such events can be communicated by sending signals propagating at the speed equal to c or less than c and so one of the two events always occurs in the past or future of the other event and thus cause and effect relation exists. Event pairs with time-like separation have a negative squared spacetime interval, i.e. $\Delta s^2 < 0$.

ii) Space-like interval

Here $\Delta r^2 > c^2\Delta t^2 \Rightarrow \Delta s^2 > 0$. If two events are separated by space-like interval, **not** enough time passes between their occurrences for there to exist a causual relationship crossing the spatial distance by the signal to communicate between the two events at the speed of light or slower. The spacetime interval for two events occur simultaneously at two different points in a reference frame is space-like. No signal moving with speed equal to c or less than c can communicate between two simultaneous events and so causal relationship does not exists. Such events are considered not to occur in each other's future or past. Event pairs with space-like separation have a positive squared spacetime interval, i.e. $\Delta s^2 > 0$.

iii) Light-like interval

Here $c^2\Delta t^2 = \Delta r^2 \;\Rightarrow \Delta s^2 = 0$. In light-like interval, the events define a squared spacetime interval of zero. Events which occur to or by a photon along its path, all have light-like separation. Cause and effect relation for such events exists as they can be communicated by sending signals at the speed of light c.

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Relativistic Doppler Effect

The increase in pitch/frequency of sound/**light** when the source approaches us or we approach the source and the decrease in pitch/frequency of sound/**light** when the source recedes from us or we recede from the source constitute the Doppler Effect/ **Relativistic Doppler Effect**.

In case of sound the relation between the frequency ν_0 of the emitted sound at source and that of ν for the received sound at the listener is given by

$$\nu = \nu_0 \frac{1 + V_l/V}{1 - V_s/V}$$

Where $V_l, V_s \, and V$ are the speeds of the listener, source and the sound w.r. to the medium (e.g. air). If anyone moves towards the other, its speed is taken as +ve and if recedes from the other, it is taken as –ve speed and if at rest w.r. to the medium, the speed is zero.

Can you apply the above equation for light?

Simply no, as the Doppler Effect in sound appears to violate the Principle of Relativity. It is because the effect in case of sound counts the individual velocities of the source and listener w.r. to the medium. But, **in case of light no medium is involved and only relative motion between the light source and the observer is meaningful.** The Doppler Effect in light must therefore differ from that in sound.

Doppler Effect in light

We analyze the Doppler Effect in light in three different situations:

1) Transverse Doppler Effect
2) Longitudinal Doppler Effect for Receding
3) Longitudinal Doppler Effect for approaching

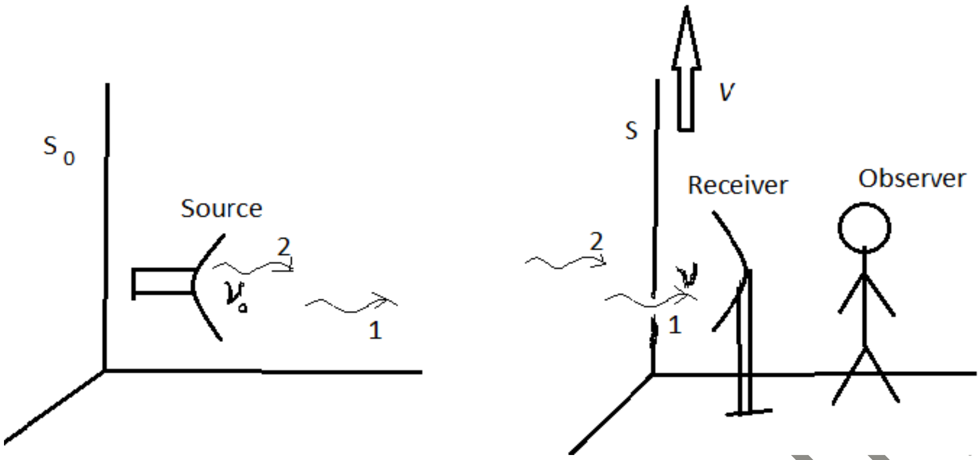
Let us discuss one by one.

1) Transverse Doppler Effect:

We consider a light source as a clock that ticks ν_0 times per second and emits a wave of light in each tick in the S_o inertial frame. So the frequency of the wave at source is ν_0 and the proper time interval between two consecutive ticks is

$$t_0 = \frac{1}{\nu_0} \text{ ....(1)}$$

and hence it is the time interval between two consecutive waves at the time of emission at source.



Now we suppose that another inertial frame S is moving with uniform speed V perpendicularly to the line joining the S frame to S_o frame. The observer who is at rest in S frame is receiving the waves approaching him from the source at rest in S_o frame. The time interval between two consecutive received waves in S frame never be equal to t_0, but it will be longer than t_0. It is because S frame is the rest frame for the observer and time is going with the fastest rate there, while S_o frame is the moving frame for him and so time is going slow in S_o frame according to the time dilation theory of Einstein's Relativity. Thus the time interval between two consecutive received waves in S frame w.r. to the observer will be

$$t = \frac{t_0}{\sqrt{1-\frac{V^2}{c^2}}} \quad \text{.......................................(2)}$$

$$\left(\text{since } \frac{V}{c} < 1, \quad \sqrt{1-\frac{V^2}{c^2}} < 1 \text{ and hence } t > t_0\right)$$

And hence number of received waves per second is $\frac{1}{t}$, which is the frequency ν of the received waves in S frame. Thus

$$\nu = \frac{1}{t} = \frac{1}{t_0}\sqrt{1-\frac{V^2}{c^2}}$$

$$\therefore \nu = \nu_0\sqrt{1-\frac{V^2}{c^2}} \quad \text{.......................................(3)}$$

Since $\sqrt{1-\frac{V^2}{c^2}} < 1 \qquad \Rightarrow \nu < \nu_0$

i.e. the observed frequency is lower than that at the source. In terms of wave length, the observed wave length λ is longer than that λ_o of the light at the source, since

Speed of light in free space $c = \lambda_o \nu_0 = \lambda \nu$ ....(4)

and hence

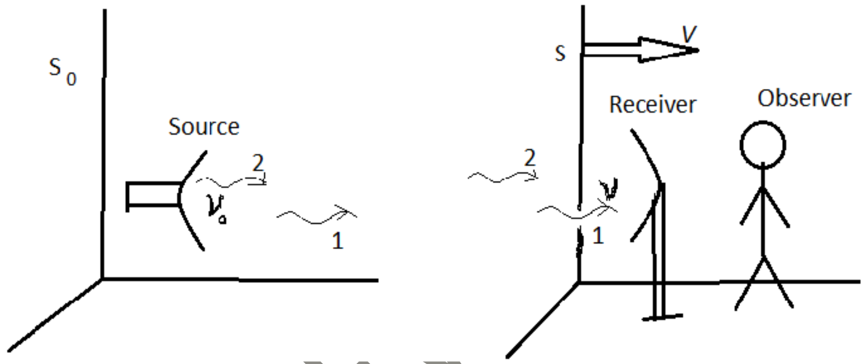
$$\lambda = \frac{\lambda_o}{\sqrt{1 - \frac{V^2}{c^2}}} \text{ ....(5)}$$

$$\Rightarrow \lambda > \lambda_o$$

As the wavelength received gets elongated, it is called **red shift** or it is said that the waves get red shifted.

2) Longitudinal Doppler Effect for Receding

a) When the observer is receding from the source:



We consider a light source as a clock that ticks ν_0 times per second and emits a wave of light in each tick in the S_o inertial frame. So the frequency of the wave at source is ν_0 and the proper time interval between two consecutive ticks is

$$t_0 = \frac{1}{\nu_0} \qquad \text{(same as eqn (1))}$$

which is the time interval between two consecutive waves at the time of emission at source.

This time we suppose that the inertial frame S is receding from inertial frame S_o with uniform speed V along the line of joining the S frame to S_o frame. Since the observer is at rest, time is going with the fastest rate there and so the time interval between two consecutive received waves in S frame is longer than t_0. According to the time dilation theory in Einstein's Relativity

$$t = \frac{t_0}{\sqrt{1 - \frac{V^2}{c^2}}} \qquad \text{(same as eqn (2))}$$

But if we look into the situation, we have found that the total time T elapsed between two consecutive received waves should be longer that t by an amount equal to $\frac{Vt}{c}$, i.e.

$$T = t + \frac{Vt}{c} = t\left(1+\frac{V}{c}\right) \text{..............................}(6)$$

It is because the later wave has to cover an **additional distance** equal to Vt in comparison to the distance travelled by the previous one. (From the time of receipt of a wave by the observer at rest in frame S, the frame S has travelled a distance Vt within t time interval before the observer is going to receive the next.)

Hence the number of received waves per second is $\frac{1}{T}$, which is the frequency ν of the received waves in S frame. Thus

$$\nu = \frac{1}{T}$$

Using eqns (1), (2) & (6) in the above eqn, we have

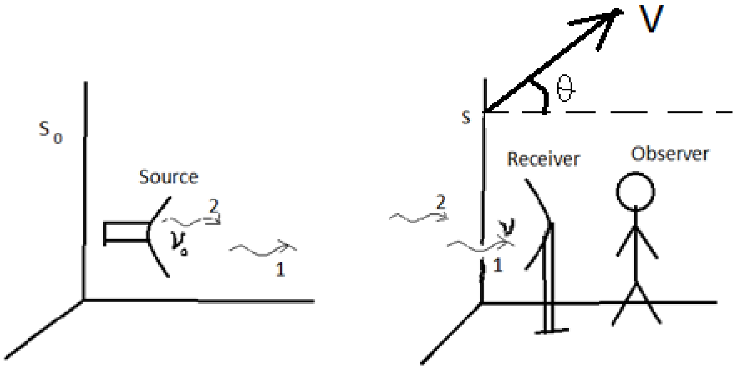
$$\nu = \nu_0\sqrt{\frac{1-V/c}{1+V/c}} \text{.................................}(7)$$

Since $\frac{V}{c} < 1$, $1-\frac{V}{c} < 1$ and $1+\frac{V}{c} > 1$ and hence $\sqrt{\frac{1-V/c}{1+V/c}} < 1 \quad \Rightarrow \nu < \nu_0$, i.e. the observed frequency is lower than that at the source. In terms of wave length, if the observed wave length be λ and that of the light at the source be λ_o, by eqns (4) and (7)

$$\lambda = \lambda_o\sqrt{\frac{1+V/c}{1-V/c}} \text{..................................}(8)$$

Since $\sqrt{\frac{1+V/c}{1-V/c}} > 1 \quad \Rightarrow \lambda > \lambda_o$, i.e. the wavelength of the received waves gets elongated or the received waves get **red shifted**.

b) **When the observer is receding from the source making an angle θ with the line joining the observer to the source:**
This time, if we look into the situation, we have found that the total time T elapsed between two consecutive received waves should be longer that t by an amount equal to $\frac{(V\cos\theta)t}{c}$, i.e.



$$T = t + \frac{(Vcos\theta)t}{c} = t\left(1 + \frac{Vcos\theta}{c}\right)$$

It is because the later wave has to cover an **additional distance** equal to $(Vcos\theta)t$ in comparison to the distance travelled by the previous one. (From the time of receipt of a wave by the observer at rest in frame S, the frame S has travelled a distance $(Vcos\theta)t$ within t time interval before the observer is going to receive the next.). Thus, the frequency of the received wave will be

$$\nu = \frac{1}{T} = \frac{1}{t\left(1 + \frac{Vcos\theta}{c}\right)} = \frac{\sqrt{1 - \frac{V^2}{c^2}}}{t_0\left(1 + \frac{Vcos\theta}{c}\right)} \Rightarrow \nu = \nu_0 \frac{\sqrt{1 - \frac{V^2}{c^2}}}{\left(1 + \frac{Vcos\theta}{c}\right)}$$

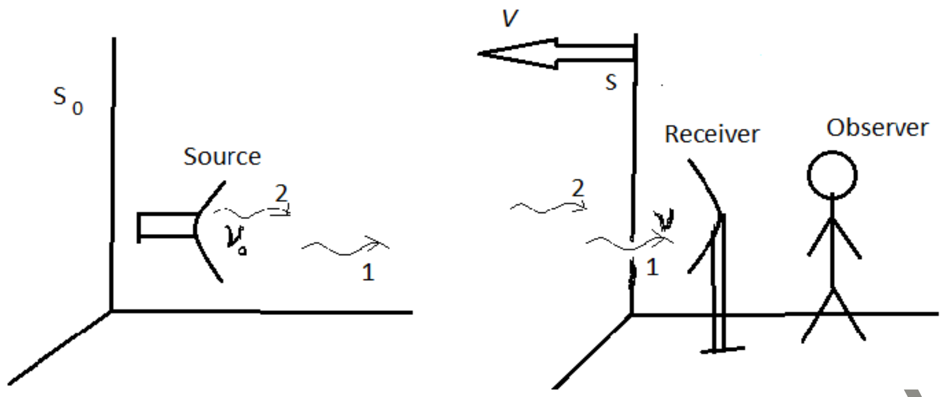
c) **When the source is receding from the observer:**

It makes no difference whether the observer is receding from the source or the source is receding from the observer, it is because only the relative velocity is meaningful in Einstein's Relativity. The S_o frame is moving away from S frame with uniform speed V along the line of joining the S frame to S_o frame. Applying the procedure same as that in 2(a), we have arrived at the eqns (7) and (8), which again show that the wavelength of the received light gets elongated or the received light gets **red shifted**.

d) **When the source is receding from the observer making an angle θ with the line joining the observer to the source:**
Same as (2b)

3) **Longitudinal Doppler Effect for approaching**

a) **When the observer is approaching the source:**
We consider a light source as a clock that ticks ν_0 times per second and emits a wave of light in each tick in the S_o inertial frame. So the frequency of the wave at source is ν_0 and the proper time interval between two consecutive ticks is



$$t_0 = \frac{1}{\nu_0} \qquad \text{(same as eqn (1))}$$

which is the time interval between two consecutive waves at the time of emission at source.

We suppose that the inertial frame S is approaching the inertial frame S_o with uniform speed V along the line of joining the S frame to S_o frame. Since the observer is at rest in S frame, time is going with the fastest rate there and so the time interval t between two consecutive received waves in S frame is longer than t_0. According to the time dilation theory in Einstein's Relativity

$$t = \frac{t_0}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \text{(same as eqn (2))}$$

But if we look into the situation, we have found that the total time T elapsed between two consecutive received waves should be shorter than t by an amount equal to $\frac{Vt}{c}$, i.e.

$$T = t - \frac{Vt}{c} = t\left(1 - \frac{V}{c}\right) \text{..............................(9)}$$

It is because the later wave has to cover a distance less by an amount equal to Vt in comparison to the distance travelled by the previous one. (From the time of receipt of a wave by the observer at rest in frame S, the frame S has travelled a distance Vt towards S_o frame within t time interval before the observer in S frame is going to receive the next.)

Hence the number of received waves per second is $\frac{1}{t}$, which is the frequency ν of the received waves in S frame. Thus

$$\nu = \frac{1}{T}$$

Using eqns (1), (2) & (9) in the above eqn, we have

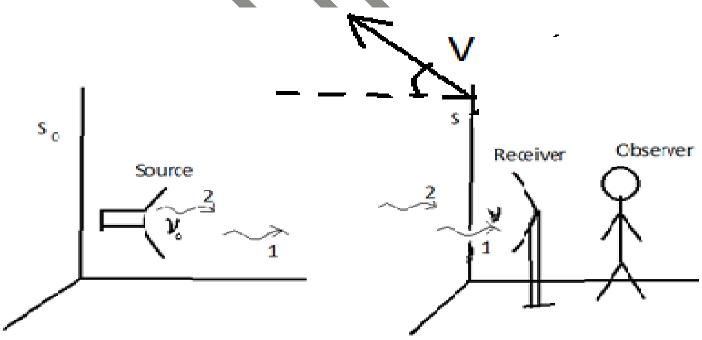
$$\nu = \nu_0 \sqrt{\frac{1+V/c}{1-V/c}} \text{ ....(10)}$$

Since $\frac{V}{c} < 1$, $1+\frac{V}{c} > 1$ and $1-\frac{V}{c} < 1$ and hence $\sqrt{\frac{1+V/c}{1-V/c}} > 1 \Rightarrow \nu > \nu_0$, i.e. the observed frequency is higher than that at the source. In terms of wave length, if the observed wave length be λ and that of the light at the source be λ_o, by eqns (4) and (10)

$$\lambda = \lambda_o \sqrt{\frac{1-V/c}{1+V/c}} \text{ ....(11)}$$

Since $\sqrt{\frac{1-V/c}{1+V/c}} < 1 \Rightarrow \lambda < \lambda_o$, i.e. the wavelength of the received waves is shorter than that of the received waves or it is said that the received waves get **blue shifted**.

b) When the observer is approaching the source making an angle θ with the line joining the observer to the source:



The total time T elapsed between two consecutive received waves should be shorter than t by an amount equal to $\frac{(V\cos\theta)t}{c}$, i.e.

$$T = t - \frac{(V\cos\theta)t}{c} = t\left(1 - \frac{V\cos\theta}{c}\right)$$

It is because the later wave has to cover a distance less by an amount equal to $(V\cos\theta)t$ in comparison to the distance travelled by the previous one. Thus, the frequency of the received wave will be

$$\nu = \frac{1}{T} = \frac{1}{t\left(1 - \frac{V\cos\theta}{c}\right)} = \frac{\sqrt{1-\frac{V^2}{c^2}}}{t_0\left(1 - \frac{V\cos\theta}{c}\right)} \Rightarrow \nu = \nu_0 \frac{\sqrt{1-\frac{V^2}{c^2}}}{\left(1 - \frac{V\cos\theta}{c}\right)}$$

c) **When the source is approaching the observer**

It makes no difference whether the observer is approaching the source or the source is approaching the observer, it is because only the relative velocity is meaningful in Einstein's Relativity. The S_o frame is moving towards the S frame with uniform speed V along the line of joining the S frame to S_o frame. Applying the procedure same as that in 3(a), we have arrived at the eqns (10) and (11), which again show that the wavelength of the received light gets contracted or the received light gets **blue shifted**.

d) **When the source is approaching the observer making an angle θ with the line joining the observer to the source:**

(same as 3b)

Problem:

A distant galaxy in the constellation Hydra is receding from the earth at $6.12 \times 10^7 m/s$. By how much is a green spectral line of wavelength 500nm (1nm=$10^{-9}m$) emitted by this galaxy shifted towards the red end of the spectrum?

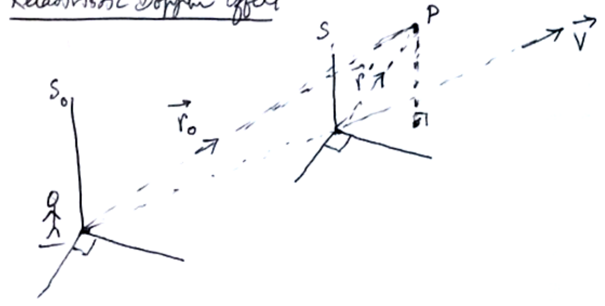
(Hints: $\lambda = \lambda_o \sqrt{\dfrac{1+V/c}{1-V/c}}$, $\frac{V}{c} = 0.204$, $\lambda = 500\sqrt{\frac{1+0.204}{1-0.204}} = 615nm$, which is in the orange part of the spectrum. The shift is $d\lambda = \lambda - \lambda_0 = 115nm$.)

How Longitudinal effect is a special case of transverse effect?

The transverse Effect includes all directions, whereas the longitudinal effect includes the motion along the line joining the source with the observer (0^0 and 180^0). The longitudinal formula can be derived mathematically from the transverse formula restricting it to 0^0 and 180^0 .

General Treatment for Relativistic Doppler Effect:

Relativistic Doppler Effect



- Phase of a plane wave $\phi = \omega t - \vec{k}\cdot\vec{r} = \omega' t' - \vec{k}'\cdot\vec{r}'$ → which is an invariant.
- $\underbrace{\omega_0 t_0 - \vec{k}_0\cdot\vec{r}_0}_{\text{In } S_0} = \underbrace{\omega t - \vec{k}\cdot\vec{r}}_{\text{In } S} \longrightarrow$ (A)
- All the vectors $\vec{k}$, $\vec{r}$, etc. are resolved into rectangular components along and $\perp^r$ to $\vec{V}$ as –

$\vec{k} = k\hat{n}$

$\vec{k}_0 = k_0\hat{n}_0 = k_0(\vec{n}_{0\parallel} + \vec{n}_{0\perp}) \longrightarrow \vec{k} = k(\vec{n}_\parallel + \vec{n}_\perp)$

$\vec{r}_0 = \vec{r}_{0\parallel} + \vec{r}_{0\perp} \longrightarrow \vec{r} = \vec{r}_\parallel + \vec{r}_\perp$

$= \gamma(\vec{r}_{0\parallel} - \vec{\beta}ct_0) + \vec{r}_{0\perp}$

$\omega = 2\pi\nu = 2\pi\frac{c}{\lambda} = ck$

$\vec{\beta} = \frac{\vec{V}}{c}$, $\vec{V}$ → uniform vel. of S w.r.t. S_0

(A) ⇒ $ck_0t_0 - k_0\hat{n}_0\cdot\vec{r}_0 = ckt - k\hat{n}\cdot\vec{r}$

$\Rightarrow k_0\left[ct_0 - (\vec{n}_{0\parallel}\cdot\vec{r}_{0\parallel} + \vec{n}_{0\perp}\cdot\vec{r}_{0\perp})\right]$

$= k\left[\gamma(ct_0 - \vec{\beta}\cdot\vec{r}_0) - (\vec{n}_\parallel\cdot\vec{r}_\parallel + \vec{n}_\perp\cdot\vec{r}_\perp)\right]$

$= k\left[\gamma\{ct_0 - (\vec{\beta}\cdot\vec{r}_{0\parallel} + \vec{\beta}\cdot\vec{r}_{0\perp})\} - \{\vec{n}_\parallel\cdot\gamma(\vec{r}_{0\parallel} - \vec{\beta}ct_0) + \vec{n}_\perp\cdot\vec{r}_{0\perp}\}\right]$

$= k\gamma\left[ct_0 - \vec{\beta}\cdot\vec{r}_{0\parallel} - \underbrace{\vec{\beta}\cdot\vec{r}_{0\perp}}_{=0} - \vec{n}_\parallel\cdot\vec{r}_{0\parallel} + \vec{n}_\parallel\cdot\vec{\beta}ct_0 - \vec{n}_\perp\cdot\vec{r}_{0\perp}\frac{1}{\gamma}\right]$

$= k\gamma\left[ct_0(1 + \vec{n}_\parallel\cdot\vec{\beta}) - (\vec{\beta} + \vec{n}_\parallel)\cdot\vec{r}_{0\parallel} - \hat{n}_\perp\cdot\vec{r}_{0\perp}\frac{1}{\gamma}\right]$

If this eqn is to hold all times t_0 and position $\vec{r}_0$, then the different coefficients for t_0, $\vec{r}_{0\parallel}$ and $\vec{r}_{0\perp}$ on either sides must be equal. That is –

$k_0 = \gamma k(1 + \vec{n}_\parallel\cdot\vec{\beta}) \xrightarrow{\omega = ck} \boxed{\omega_0 = \gamma\omega(1 + \vec{n}_\parallel\cdot\vec{\beta})}$ — (B)

$k_0\hat{n}_{0\parallel} = k\gamma(\vec{\beta} + \vec{n}_\parallel) \longrightarrow k_{0\parallel} = \gamma(k\vec{\beta} + k\vec{n}_\parallel)$

$\therefore \boxed{k_{0\parallel} = \gamma\left(\frac{\omega}{c}\vec{\beta} + k_\parallel\right)}$ — (C)

$k_0\hat{n}_{0\perp} = k\hat{n}_\perp \longrightarrow \boxed{\vec{k}_{0\perp} = \vec{k}_\perp}$ — (D)

The first eqn (B) is that for Doppler shift. The frequency of the light observed at S coming from S_0 →

$\cancel{\omega = \frac{\omega_0}{\gamma(1 + \hat{n}_\parallel\cdot\vec{\beta})}}$

Inverse

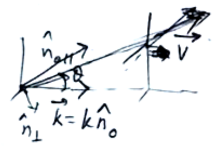
$\omega = \gamma\omega_0(1 + \vec{n}_{0\parallel}\cdot(-\vec{\beta}))$

$\nu = \nu_0\frac{1 - \vec{n}_{0\parallel}\cdot\vec{\beta}}{\sqrt{1 - \frac{V^2}{c^2}}}$

$= \nu_0\frac{1 - \frac{V\cos\theta}{c}}{\sqrt{1 - V^2/c^2}}$

$\vec{n}_\parallel\cdot\vec{\beta} = \frac{1}{c}(\vec{n}_\parallel\cdot\vec{V})$

$= \frac{V\cos\theta}{c}$



- Observer in S frame. –

$$\omega_0 t_0 - \vec{k}_0\cdot\vec{r}_0 = \omega t - \vec{k}\cdot\vec{r}$$

$$\hat{n}_0 = \hat{n}_{0\parallel} + \hat{n}_{0\perp}$$
$$\vec{r}_0 = \vec{r}_{0\parallel} + \vec{r}_{0\perp} = \gamma(\vec{r}_\parallel + \vec{\beta}ct) + \vec{r}_\perp$$

$$\Rightarrow k_0(ct_0 - \hat{n}_0\cdot\vec{r}_0) = k(ct - \hat{n}\cdot\vec{r})$$

$$\Rightarrow k_0\left[\gamma(ct + \vec{\beta}\cdot\vec{r}) - (\hat{n}_{0\parallel} + \hat{n}_{0\perp})\cdot(\vec{r}_{0\parallel} + \vec{r}_{0\perp})\right] = k\left[ct - (\hat{n}_\parallel + \hat{n}_\perp)\cdot(\vec{r}_\parallel + \vec{r}_\perp)\right]$$

$$\Rightarrow k_0\left[\gamma(ct + \vec{\beta}\cdot\vec{r}_\parallel - \cancel{\vec{\beta}\cdot\vec{r}_\perp}) - \hat{n}_{0\parallel}\cdot\vec{r}_{0\parallel} - \hat{n}_{0\perp}\cdot\vec{r}_{0\perp}\right] = k\left[ct - \hat{n}_\parallel\cdot\vec{r}_\parallel - \hat{n}_\perp\cdot\vec{r}_\perp\right]$$

$$\Rightarrow k_0\left[\gamma(ct + \vec{\beta}\cdot\vec{r}_\parallel) - \overset{=0}{\hat{n}_{0\parallel}}\cdot\gamma(\vec{r}_\parallel + \vec{\beta}ct) - \hat{n}_{0\perp}\cdot\vec{r}_\perp\right] = k\left[ct - \hat{n}_\parallel\cdot\vec{r}_\parallel - \hat{n}_\perp\cdot\vec{r}_\perp\right]$$

$$\Rightarrow k_0\gamma\left[ct + \vec{\beta}\cdot\vec{r}_\parallel - \hat{n}_{0\parallel}\cdot\vec{r}_\parallel - \hat{n}_{0\parallel}\cdot\vec{\beta}ct - \frac{1}{\gamma}\hat{n}_{0\perp}\cdot\vec{r}_\perp\right] = k\left[ct - \quad\right]$$

$$\Rightarrow k_0\gamma\left[ct(1 - \hat{n}_{0\parallel}\cdot\vec{\beta}) - (\vec{\beta} - \hat{n}_{0\parallel})\cdot\vec{r}_\parallel - \frac{1}{\gamma}\hat{n}_{0\perp}\cdot\vec{r}_\perp\right] = k\left[ct - \text{---}\right]$$

If this eqn is to hold all all times t and positions $\vec{r}$, then the different coefficients of t, $\vec{r}_\parallel$ & $\vec{r}_\perp$ on either side must be equal. So,

$$k_0\gamma(1 - \hat{n}_{0\parallel}\cdot\vec{\beta}) = k \xrightarrow{\omega = ck} \boxed{\omega_0\gamma(1 - \hat{n}_{0\parallel}\cdot\vec{\beta}) = \omega} \longrightarrow \text{Doppler Effect}$$

$$(\vec{\beta} - \hat{n}_{0\parallel})k_0\gamma = k\hat{n}_\parallel \longrightarrow \gamma\left(\frac{\omega_0}{c}\vec{\beta} - \hat{n}_{0\parallel}k_0\right) = k\hat{n}_\parallel$$

$$\therefore \boxed{\gamma\left(\frac{\omega_0}{c}\vec{\beta} - k_{0\parallel}\right) = k_\parallel}$$

$$k_0\gamma\left(\frac{1}{\gamma}\hat{n}_{0\perp}\right) = k\hat{n}_\perp \longrightarrow k_0\hat{n}_{0\perp} = k\hat{n}_\perp \Rightarrow \boxed{\vec{k}_{0\perp} = \vec{k}_\perp}$$

Transformation eqn for wavenumber

Relativistic Doppler Effect is called 2nd order effect. Why?

- The transverse Effect is sometimes called second order effect only to avert the confusion with the longitudinal effect.
- The frequency relation in Doppler Effect for sound is given by

$$\nu = \nu_0 \frac{1+{V_l}/{V}}{1-{V_s}/{V}}$$

If the source is receding from the stationary listener, $V_l = 0$ and V_s is taken as –ve speed.

Hence

$$\nu = \nu_0 \left(1+{V_s}/{V}\right)^{-1}$$

Or

$$\frac{\nu}{\nu_0} = \left(1+{V_s}/{V}\right)^{-1} = 1-\frac{V_s}{V}+\left(\frac{V_s}{V}\right)^2-\left(\frac{V_s}{V}\right)^3+\ldots\ldots\ldots\ldots$$

Applying

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots\ldots\ldots\ldots, (-1<n<1)$$

And if the source is approaching a stationary listener, then

$$\nu = \nu_0 \left(1-{V_s}/{V}\right)^{-1}$$

Or

$$\frac{\nu}{\nu_0} = \left(1-{V_s}/{V}\right)^{-1} = 1+\frac{V_s}{V}+\left(\frac{V_s}{V}\right)^2+\left(\frac{V_s}{V}\right)^3+\ldots\ldots\ldots\ldots$$

For light, if the source is receding from the observer, then by eqn (7)

$$\nu = \nu_0\sqrt{\frac{1-V/c}{1+V/c}} = \left(1-V/c\right)^{1/2}\left(1+V/c\right)^{-1/2}$$

$$\therefore \frac{\nu}{\nu_o} = \left(1-\frac{1}{2}\frac{V}{c}-\frac{1}{8}\left(\frac{V}{c}\right)^2+........\right)\left(1-\frac{1}{2}\frac{V}{c}+\frac{3}{8}\left(\frac{V}{c}\right)^2+........\right)$$

$$\frac{\nu}{\nu_o} = \left(1-\frac{V}{c}+\frac{1}{2}\left(\frac{V}{c}\right)^2+........\right)$$

For light, if the source is approaching the observer, then by eqn (9)

$$\nu = \nu_0\sqrt{\frac{1+V/c}{1-V/c}} = \left(1+V/c\right)^{1/2}\left(1-V/c\right)^{-1/2}$$

$$\therefore \frac{\nu}{\nu_o} = \left(1+\frac{1}{2}\frac{V}{c}-\frac{1}{8}\left(\frac{V}{c}\right)^2+........\right)\left(1+\frac{1}{2}\frac{V}{c}+\frac{3}{8}\left(\frac{V}{c}\right)^2+........\right)$$

$$\frac{\nu}{\nu_o} = \left(1+\frac{V}{c}+\frac{1}{2}\left(\frac{V}{c}\right)^2+........\right)$$

If we consider only the second order term in Relativistic Doppler Effect for light, then only the result is found to be different from classical Doppler Effect for sound. That is why Relativistic Doppler Effect is called the 2nd order effect.

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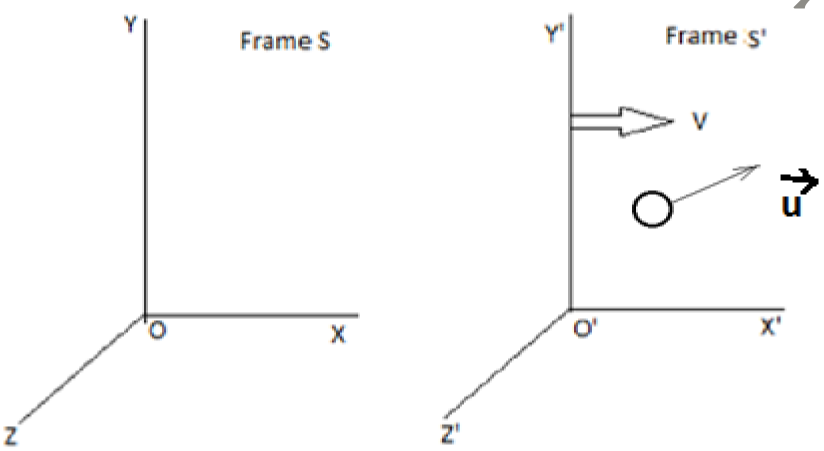
Transformation Equations for velocity, mass, momentum and energy

Velocity Addition Theorem in Einstein's Relativity

The motion of a ball in S' frame is observe by two observers from S and S' inertial frames and $\boldsymbol{u} = (u_x, u_y, u_z)$ and $\boldsymbol{u'} = (u_x{}', u_y{}', u_z{}')$ be the velocities of the ball respectively.



By definition

$$u_x = \frac{dx}{dt} \text{ and } u'_x = \frac{dx'}{dt'}\text{, etc.}$$

Applying the inverse Lorentz-Einstein transformation equations in the above definitions, we can derive the following relations between different components of velocities.

$$u_x = \frac{u'_x + \mathrm{V}}{1 + \frac{V}{c^2} u_x{}'}, \quad u_y = \frac{\left(\sqrt{1 - \frac{V^2}{c^2}}\right) u'_y}{1 + \frac{V}{c^2} u_x{}'} \quad \text{and} \quad u_z = \frac{\left(\sqrt{1 - \frac{V^2}{c^2}}\right) u'_z}{1 + \frac{V}{c^2} u_x{}'}$$

- In classical speed limit$\left(\frac{V}{c} \ll 1\right)$,

$$1 - \frac{V^2}{c^2} \approx 1 \quad \text{and} \quad \frac{V}{c^2} \approx 0\,,$$

the above velocity transformation equations revert back to Galilean type, i. e.

$$u_x = u'_x + \mathrm{V}, \quad u_y = u'_y \quad \text{and} \quad u_z = u'_z$$

It indirectly proves that the velocity transformation equations in Einstein's relativity are correct.

The transformation equations for acceleration could also be worked out using the same procedure, but they are not of particular use in relativity. In Newtonian mechanics, force can be calculated from acceleration multiplying it by mass according to the Newton's second law of motion. But this trick will not work in Einstein's relativity. Why? Because mass is no longer an absolute concept there. We will divulge that point later.

- If we imagine a photon instead of a material particle moving along the + X direction in S' frame, then it can be shown very easily by using the above velocity transformation equation that the speed of the photon w. r. to an observer in S frame is again equal to c. (Since $u_x' = c,\ u_x = \frac{c+V}{1+\frac{V}{c^2}c}$ = c) It proves that the structuring of the Lorentz-Einstein transformation equations to maintain the speed of light a constant equal to c is precisely done.

- The above calculation also depicts that **any velocity added to c yields again the c** and c is a constant independent of the relative motion of observer and source.

- If the speed of light in free space is an absolute concept in Einstein's relativity, then what would be the fate of space and time in Galilean Relativity (or Newtonian mechanics)? Would they remain as two absolute concepts? What about the mass?

Nothing can moves faster than light

If we suppose that the speed of the material body in S' frame along +X direction and the speed of the frame itself are very near to c, then the velocity transformation equation in Einstein's relativity shows that the speed w. r. to the observer in S frame does not exceed c which contradicts the conclusion drawn from Newtonian mechanics. Let us choose $u_x{}' = c - \delta$ and $V = c - \delta$, here δ is very small. By using transformation rule

$$u_x = \frac{(c-\delta)+(c-\delta)}{1+\frac{(c-\delta)}{c^2}(c-\delta)} = \frac{2(c-\delta)c^2}{2c(c-\delta)+\delta^2}$$

$$u_x < \frac{2(c-\delta)c^2}{2c(c-\delta)} = c$$

According to Newtonian mechanics

$$u_x = u_x' + \mathrm{V} = 2c - 2\delta \; > c$$

Transformation equations of acceleration

The motion of a body moving with velocity $\vec{u'}$ $(\vec{u'} = \hat{i}u_x' + \hat{j}u_y' + \hat{k}u_z')$ in S' frame is observed by two observers from S' frame itself and from another inertial frame S. Let S' frame is moving with uniform speed V along $+X$ direction w.r. to S frame. During the motion of frame S', X' axis remains coincident with X axis and Y' remains parallel with Y and Z' axis with Z axis. If $\vec{u}$ $(\vec{u} = \hat{i}u_x + \hat{j}u_y + \hat{k}u_z)$ be the velocity of the body as observed by the observer from S frame, then by applying the Lorentz-Einstein transformation equations

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}, \qquad y' = y, \qquad z' = z, \qquad t' = \frac{t - \frac{V}{c^2}x}{\sqrt{1 - \frac{V^2}{c^2}}},$$

And also applying the definition of velocity, i.e. $u_x' = \frac{dx'}{dt'} = \frac{\frac{dx'}{dt}}{\frac{dt'}{dt}}$, etc., we can derive the following velocity addition theorem in Einstein's Relativity.

$$u_x' = \frac{u_x - V}{1 - \frac{V}{c^2}u_x}, \qquad u_y' = \left(\frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V}{c^2}u_x}\right)u_y, \qquad u_z' = \left(\frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V}{c^2}u_x}\right)u_z.$$

If we suppose that the body is moving with acceleration in frame S', and observe acceleration is $\vec{a} = (a_x, a_y, a_z)$ w.r. to frame S and $\vec{a'} = (a_x', a_y', a_z')$ w.r. to frame S'. By definition

$$a_x = \frac{du_x}{dt}, \qquad a_y = \frac{du_y}{dt}, \qquad a_z = \frac{du_z}{dt},$$

And $$a_x' = \frac{du_x'}{dt'} \qquad a_y' = \frac{du_y'}{dt'} \qquad a_z' = \frac{du_z'}{dt'}$$

$$a'_x = \frac{du'_x}{dt'} = \frac{\frac{du_x}{dt}}{\frac{dt'}{dt}} = \frac{\frac{a_x - 0}{1 - \frac{V}{c^2}u_x} - \frac{u_x - V}{\left(1 - \frac{V}{c^2}u_x\right)^2}\left(0 - \frac{V}{c^2}a_x\right)}{\frac{1 - \frac{V}{c^2}u_x}{\sqrt{1 - \frac{V^2}{c^2}}}}$$

After simplifying

$$a'_x = \frac{\left(1 - \frac{V^2}{c^2}\right)^{\frac{3}{2}}}{\left(1 - \frac{V}{c^2}u_x\right)^3} a_x$$

And $a'_y = \frac{du'_y}{dt'} = \frac{\frac{du'_y}{dt}}{\frac{dt'}{dt}} = \frac{\left(\frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V}{c^2}u_x}\right) a_y - \frac{\sqrt{1 - \frac{V^2}{c^2}}}{\left(1 - \frac{V}{c^2}u_x\right)^2}\left(0 - \frac{V}{c^2}a_x\right) u_y}{\frac{1 - \frac{V}{c^2}u_x}{\sqrt{1 - \frac{V^2}{c^2}}}}$

After simplifying

$$a'_y = \frac{\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{V}{c^2}u_x\right)^2}\left[a_y + \frac{\frac{V}{c^2}}{\left(1 - \frac{V}{c^2}u_x\right)} u_y a_x\right]$$

Similarly

$$a'_z = \frac{\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{V}{c^2}u_x\right)^2}\left[a_z + \frac{\frac{V}{c^2}}{\left(1 - \frac{V}{c^2}u_x\right)} u_z a_x\right]$$

Transformation properties (equations of mass)

The motion of a body (of rest mass m_0) moving with velocity $\vec{u'}$ in S' frame is observed by two observers from S' frame itself and from another inertial frame S. Let S' frame is moving with uniform speed V along $+X$ direction w.r. to S frame. During the motion of frame S', X' axis remains coincident with X axis and Y' remains parallel with Y and Z' axis with Z axis. If $\vec{u}$ be the velocity of the body as observed by the observer from S frame and if m be its relativistic mass in S frame and m' be the relativistic mass for the same body moving with velocity $\vec{u'}$ in S' frame, then by the mass variation theorem in Einstein's Relativity

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \qquad \text{and} \qquad m' = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}},$$

where $\vec{u} = \hat{i}u_x + \hat{j}u_y + \hat{k}u_z \qquad u^2 = u_x^2 + u_y^2 + u_z^2$

$$\vec{u'} = \hat{i}u'_x + \hat{j}u'_y + \hat{k}u'_z \qquad u'^2 = u_x'^2 + u_y'^2 + u_z'^2$$

Thus the transformation relation for mass will be

$$m' = \frac{\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u'^2}{c^2}}}\, m.$$

We are now going to replace $\sqrt{1-\frac{u'^2}{c^2}}$ in S' frame appeared in RHS in the above transformation relation with an expression expressed in S frame.

By applying the Lorentz-Einstein transformation equations

$$x' = \frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad y' = y, \qquad z' = z, \qquad t' = \frac{t - \frac{V}{c^2}x}{\sqrt{1-\frac{V^2}{c^2}}},$$

And also applying the definition of velocity, i.e. $u'_x = \frac{dx'}{dt'} = \frac{\frac{dx'}{dt}}{\frac{dt'}{dt}}$,etc., we can derive the following velocity addition theorem in Einstein's Relativity.

$$u'_x = \frac{u_x - V}{1 - \frac{V}{c^2}u_x}, \qquad u'_y = \left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}u_x}\right)u_y, \qquad u'_z = \left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}u_x}\right)u_z .$$

To evaluate $\left(1-\frac{u'^2}{c^2}\right)$, we follow the mathematics given below.

$$c^2 - u'^2 = c^2 - \left(u'^2_x + u'^2_y + u'^2_z\right)$$

$$= c^2 - \left[\left(\frac{u_x - V}{1-\frac{V}{c^2}u_x}\right)^2 + \left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}u_x}u_y\right)^2 + \left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}u_x}u_z\right)^2\right]$$

$$= c^2 - \frac{1}{\left(1-\frac{V}{c^2}u_x\right)^2}\left[\left(u_x - V\right)^2 + \left(1-\frac{V^2}{c^2}\right)u_y^2 + \left(1-\frac{V^2}{c^2}\right)u_z^2\right]$$

$$= c^2 - \frac{1}{\left(1-\frac{V}{c^2}u_x\right)^2}\left[u^2 + V^2 - 2u_xV - \frac{V^2}{c^2}\left(u_y^2 + u_z^2\right)\right]$$

$$= \frac{1}{\left(1-\frac{V}{c^2}u_x\right)^2}\left[c^2\left(1-\frac{V}{c^2}u_x\right)^2 - \left\{u^2 + V^2 - 2u_xV - \frac{V^2}{c^2}\left(u^2 - u_x^2\right)\right\}\right]$$

After simplifying

$$1-\frac{u'^2}{c^2} = \frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V}{c^2}u_x\right)^2}\left(1-\frac{u^2}{c^2}\right)$$

Using it in the transformation relation for mass, we have

$$m' = \frac{1-\frac{V}{c^2}u_x}{\sqrt{\left(1-\frac{V^2}{c^2}\right)}}m .$$

- If the body at rest in S frame, $u_x = 0$ and $m = m_0$, then the mass of the body moving with uniform speed V along $-X$ direction w.r.t. to the observer in S' frame becomes

$$m' = \frac{m_0}{\sqrt{\left(1-\frac{V^2}{c^2}\right)}},$$

which is in exact agreement with the mass variation formula in Einstein's Relativity.

Transformation properties (equations) of momentum

The motion of a body (of rest mass m_0) moving with speed u' in S' frame is observed by two observers from S' frame itself and from another inertial frame S. Let S' frame is moving with uniform speed V along $+X$ direction w.r.t. to S frame. During the motion of frame S', X' axis remains coincident with X axis and Y' remains parallel with Y and Z' axis with Z axis. If u be the speed of the body as observed by the observer from S frame and if m be its relativistic mass in S frame and m' be the relativistic mass for the same body moving with speed u' in S' frame, then by the mass variation theorem in Einstein's Relativity

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \quad \text{and} \quad m' = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}},$$

where $\vec{u} = \hat{i}u_x + \hat{j}u_y + \hat{k}u_z \qquad u^2 = u_x^2 + u_y^2 + u_z^2$

$\vec{u'} = \hat{i}u'_x + \hat{j}u'_y + \hat{k}u'_z \qquad u'^2 = u'^2_x + u'^2_y + u'^2_z$

Momentum of the body w.r.t. to S and S' frames

$$\vec{p} = \left(p_x, p_y, p_z\right) \qquad \text{and} \qquad \vec{p'} = \left(p'_x, p'_y, p'_z\right)$$

where $p_x = mu_x = \dfrac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}u_x, \qquad p_y = mu_y = \dfrac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}u_y, \qquad p_z = mu_z = \dfrac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}u_z$,

and

$$p'_x = mu'_x = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}u'_x, \qquad p'_y = mu'_y = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}u'_y, \qquad p'_z = mu'_z = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}u'_z.$$

By applying the Lorentz-Einstein transformation equations

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}, \qquad y' = y, \qquad z' = z, \qquad t' = \frac{t - \frac{V}{c^2}x}{\sqrt{1 - \frac{V^2}{c^2}}},$$

and also applying the definition of velocity, i.e. $u'_x = \frac{dx'}{dt'} = \frac{\frac{dx'}{dt}}{\frac{dt'}{dt}}$,etc., we can derive the following velocity addition theorem in Einstein's Relativity.

$$u'_x = \frac{u_x - V}{1 - \frac{V}{c^2}u_x}, \qquad u'_y = \left(\frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V}{c^2}u_x}\right)u_y, \qquad u'_z = \left(\frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V}{c^2}u_x}\right)u_z.$$

Thus, the components of momentum vector $\overrightarrow{p'}$ for the moving body in S' frame are

$$p'_x = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}}\frac{u_x - V}{1 - \frac{V}{c^2}u_x}, \qquad p'_y = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}}\left(\frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V}{c^2}u_x}\right)u_y, \qquad p'_z = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}}\left(\frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V}{c^2}u_x}\right)u_z.$$

We are now going to replace $\sqrt{1 - \frac{u'^2}{c^2}}$ in S' frame appeared in RHS in the above equations with an expression expressed in S frame.

To evaluate $\left(1 - \frac{u'^2}{c^2}\right)$, we follow the mathematics given below.

$$c^2 - u'^2 = c^2 - \left(u'^2_x + u'^2_y + u'^2_z\right)$$

(already done)

$$= \frac{1}{\left(1 - \frac{V}{c^2}u_x\right)^2}\left[c^2\left(1 - \frac{V}{c^2}u_x\right)^2 - \left\{u^2 + V^2 - 2u_xV - \frac{V^2}{c^2}\left(u^2 - u_x^2\right)\right\}\right]$$

After simplifying

$$1-\frac{u'^2}{c^2}=\frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V}{c^2}u_x\right)^2}\left(1-\frac{u^2}{c^2}\right)$$

Applying it along with the mass-energy equivalence principle $E=mc^2$, mass variation formula $m=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$ and expressions of p_x, p_y, p_z in the above expressions of components p'_x, p'_y, p'_z, we have

$$p'_x=\frac{p_x-\frac{V}{c^2}E}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad p'_y=p_y, \qquad p'_z=p_z$$

The inverse of these transformation relations can be obtained easily by changing the primed by unprimed and unprimed by primed quantities and V by $-V$, i.e. $\left(\vec{p},E\right) \rightleftharpoons \left(\vec{p'},E'\right)$ and $V \rightleftharpoons -V$.

$$p_x=\frac{p'_x+\frac{V}{c^2}E'}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad p_y=p'_y, \qquad p_z=p'_z.$$

Transformation properties (equations) for Energy

The motion of a body (of rest mass m_0) moving with speed u' in S' frame is observed by two observers from S' frame itself and from another inertial frame S. Let S' frame is moving with uniform speed V along $+X$ direction w.r. to S frame. During the motion of frame S', X' axis remains coincident with X axis and Y' remains parallel with Y and Z' axis with Z axis. If u be the speed of the body as observed by the observer from S frame and if m be its relativistic mass in S frame and m' be the relativistic mass for the same body moving with speed u' in S' frame, then by the mass variation theorem in Einstein's Relativity

$$m=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \qquad \text{and} \qquad m'=\frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}},$$

where $\vec{u}=\hat{i}u_x+\hat{j}u_y+\hat{k}u_z \qquad u^2=u_x^2+u_y^2+u_z^2$

$$\vec{u'} = \hat{i}u'_x + \hat{j}u'_y + \hat{k}u'_z \qquad u'^2 = u'^2_x + u'^2_y + u'^2_z$$

By the mass-energy equivalence principle $E = mc^2$,

Energy of the body w.r. to frame S $\qquad E = mc^2 = \dfrac{m_0}{\sqrt{1-\dfrac{u^2}{c^2}}}c^2$

Energy of the body w.r. to frame S' $\qquad E' = m'c^2 = \dfrac{m_0}{\sqrt{1-\dfrac{u'^2}{c^2}}}c^2$

Hence, energy transformation equation

$$E' = \frac{\sqrt{1-\dfrac{u^2}{c^2}}}{\sqrt{1-\dfrac{u'^2}{c^2}}}E$$

We are now going to replace $\sqrt{1-\dfrac{u'^2}{c^2}}$ in S' frame appeared in RHS in the above equations with an expression expressed in S frame.

To evaluate $\left(1-\dfrac{u'^2}{c^2}\right)$, we follow the mathematics given below.

$$c^2 - u'^2 = c^2 - \left(u'^2_x + u'^2_y + u'^2_z\right)$$

(already done)

$$= \frac{1}{\left(1-\dfrac{V}{c^2}u_x\right)^2}\left[c^2\left(1-\frac{V}{c^2}u_x\right)^2 - \left\{u^2 + V^2 - 2u_xV - \frac{V^2}{c^2}\left(u^2 - u_x^2\right)\right\}\right]$$

After simplifying

$$1-\frac{u'^2}{c^2} = \frac{\left(1-\dfrac{V^2}{c^2}\right)}{\left(1-\dfrac{V}{c^2}u_x\right)^2}\left(1-\frac{u^2}{c^2}\right)$$

Applying it in energy transformation relation

$$E' = \left(\frac{\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{V}{c^2} u_x\right)^2} \left(1 - \frac{u^2}{c^2}\right) \right)^{-\frac{1}{2}} \sqrt{1 - \frac{u^2}{c^2}} E \quad = \frac{\left(1 - \frac{V}{c^2} u_x\right)}{\sqrt{1 - \frac{V^2}{c^2}}} E$$

Using $E = mc^2$ and $p_x = mu_x$, the final transformation relation for energy is obtained as below.

$$E' = \frac{E - p_x V}{\sqrt{1 - \frac{V^2}{c^2}}}$$

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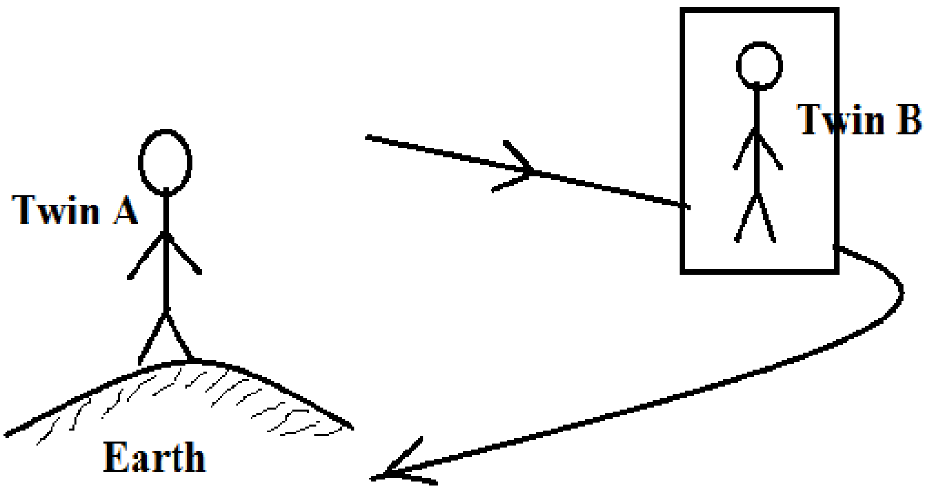
Twin Paradox

We consider two twins A and B and twin B is boarded on a spaceship for a round trip space voyage, while twin A is at rest on Earth. The spaceship of twin B could fly with the speed close to the speed of light. When twin B got departed from A, he synchronized his clock with the clock of his brother A. **According to twin A**, since B is moving away from him with very high speed, his clock must go slow. Therefore, after completion of the trip of B, when A would meet his twin brother B, he would find that he (twin A) would get older than B (i.e. A older, B younger).

If we discuss the ageing of twins A and B from the point of view of B, we will run into an apparent paradox. **According to twin B**, the clock of A must go slow since A is moving away from him with very high speed and so ageing of A would be slower compared to that of B. Therefore, after completion of the trip of A, when B would meet his twin brother A, he would find himself older than A (i.e. B older, A younger). Thus, when the twin B ends up his journey, both the twins A and B find themselves to be older from their own point of view. But, **at a time**, they never be older or younger and that is the paradox.



***Note**: The twin paradox actually is not a paradox, as it can be resolved by noting the motions of the twins A and B which are found to be asymmetric. Twin A always is at rest on Earth, whereas twin B completes a round trip and comes back to Earth. When the spaceship of twin B is taking the turn to reverse the direction of its motion towards Earth, the spaceship is behaving like an accelerated frame (so non-inertial) and as a result of that twin B inside the spaceship is experiencing some forces similar to those experienced by a passenger inside the moving bus at the turning . But, twin A feels nothing and the overall effect will be such that the twin A on Earth gets older.*

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Concept of Ether and Michelson-Morley Experiment

1. What is **ether** and **ether wind**?
2. Discuss why the idea of all-pervading medium called ether was introduced.
3. Describe the **Michelson-Morley Experiment**.
4. Point out the **logical conclusion** of Michelson-Morley Experiment.
   Or, What important **conclusion** can be drawn from Michelson-Morley Experiment?
5. How did Michelson interpret the **negative result** of his experiment?
6. How did Einstein interpret the negative result of Michelson-Morley Experiment?
7. Show how the result of Michelson-Morley Experiment supports Einstein's Postulates of special Relativity. (1995)

Postulates of Special Relativity

1. State the **two postulates of Special Relativity**. (2009)
2. Einstein gave both the theory of special relativity and the theory of photo-electric effect in 1905. In former theory he banished ether from Physics. Comment on whether it is possible to come to the same conclusion from the later theory also. (1999)
3. Write a short note on **Reasoning leading to the two postulates of Special Relativity**. (1999)
4. What is the **limit** where Special Relativity goes to the Newtonian Relativity
   Discuss how the laws of electromagnetism leads to the relativistic principle. (2009)

Lorentz Transformation and its consequences

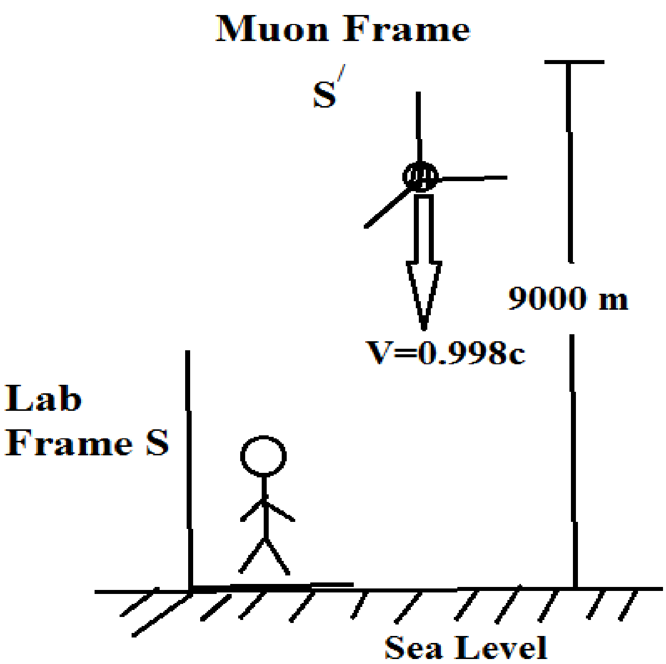
1. Derive **Lorentz space time transformation equations** for two inertial frames.
2. State the condition under which the Lorentz-Einstein Transformation is **relevant**.
3. What is a speed of space craft whose clock runs 1 second slow per hour relative to a clock on the earth?
4. Explain how the transformation equation relating the length of a rod at rest to its length in motion indicates that the free space velocity is the upper limit of all velocities. (1994)
5. From Lorentz-Einstein Transformation equations, explain
   (a) **Relativity in Simultaneity**
   (b) **Length contraction** and
   (c) **Time dilation** (1998)
6. Define **proper time**.
7. Write the **consequences** of Lorentz Transformation.
8. Is it true that two events which occur at the same place and same time for one observer will be simultaneous for all observers? Explain.
9. Find the Lorentz transformation expression for **'area'** and **'volume'.**
10. Describe **Twin Paradox** of Special Theory of Relativity.
11. Write short note on **Ultimate Speed**.

12. Show how Lorentz-Einstein Transformation equations can be used to derive the formulas of **transformation of velocities**.
    Or, Derive the **velocity addition theorem** in Special Relativity.
13. Show that the velocity addition theorem is consistent with the second postulate of Special Relativity.
14. Using velocity transformation equation, show that the velocity of light in vacuum is the same in any two systems in uniform relative motion.
15. Show that any velocity (less than c) relativistically added to c gives a result c.
    (Hints: Let, the speed of the frame S$^{/}$ w.r. to S frame S is $V = c - \epsilon$ and a particle (photon) is moving with speed equal to $u_x^{/} = c$ along X-axis in S$^{/}$ frame. Applying relativistic velocity addition theorem, the speed of the photon from S frame is found to be-
    $$u_x = \frac{u_x^{/}+V}{1+\frac{V}{c^2}u_x^{/}} = \frac{c+(c-\epsilon)}{1+\frac{c-\epsilon}{c^2}c} = c)$$
16. Muons have a mean life time of 2 μs. Cosmic ray muons are created at an altitude of 9000m and travel towards the earth surface at a speed of 0.998c, where c is the speed of light in free space. Apply the relativistic concepts of
    (i) length contraction and
    (ii) time dilation

    to show that it is possible for the muons to reach the sea level before decaying.

(Hints:

Classical physics: Distance travelled before decaying $l = V \times t = 0.998c \times 2\ \mu s = 598m$, so muons can not reach earth's surface before they undergo decay.



Using the concept of time dilation

Mean life time w.r.t the S frame on earth $\Delta t = \dfrac{\Delta t'}{\sqrt{1-\dfrac{V^2}{c^2}}} = \dfrac{2\ \mu\text{s}}{\sqrt{1-\dfrac{(0.998c)^2}{c^2}}} = 31.64\mu\text{s}$,

hence distance travel before decaying $l = V \times t = 0.998\text{c} \times 31.64\mu\text{s}{=}9473\text{m}$. Thus they can reach earth surface before decay.

Using the concept of length contraction:

Distance travelled by the muons in lab. frame w.r.t muon frame to reach earth surface

$$\text{l}{=}\text{l}_0\sqrt{1-\frac{V^2}{c^2}} = 9000\times\sqrt{1-\frac{(0.998c)^2}{c^2}} = 508.8m\ .$$

Thus time require by the muons to reach the earth surface

$$\Delta t' = \frac{l'}{V} = \frac{508.8m}{0.998c} = 1.7\mu\text{s}$$

which is shorter than their mean life time and so they can reach earth's surface before decay.)

17. The lifetime of an unstable particle at rest is 10^{-3}s. If the instant of creation, it moves with a creation, it moves with a speed of 0.9c, what is the distance it will traverse before decaying. (1991)
18. Pions are radioactive and when they are brought to rest, their half-life is measured to be 1.77×10^{-8}s. A collimated beams of pions moving at a speed of 0.99c is found to drop to half of its original intensity 39m from accelerator where they are produced. Explain this result in terms of either time dilation or length contraction. (1995) 5
(Hints:
Classically:
Distance travelled just before dropping the beam intensity to half of its original intensity $d = V \times T_{1/2} = (0.99c)\times\left(1.77\times10^{-8}s\right) = 5.26m$
Relativistically:
(a) using the concept of length contraction:
The contracted length $d' = d\sqrt{1-\dfrac{V^2}{c^2}} = 39\times\sqrt{1-\dfrac{(0.99c)^2}{c^2}} = 5.50m$
(b)using the concept of time dilation: Half life time for the pions w. r. to the laboratory frame
$$T_{1/2} = \frac{T'_{1/2}}{\sqrt{1-\dfrac{V^2}{c^2}}} = \frac{1.77\times10^{-8}}{\sqrt{1-\dfrac{(0.99c)^2}{c^2}}} = 12.55\times10^{-8}\text{s}$$

Hence distance travelled before dropping the beam intensity to half of its original intensity $d = V \times T_{1/2} == (0.99c) \times (12.55 \times 10^{-8} s) = 37.27m$)

19. The speed of a beam of particles which have a half life of 2×10^{-6}s is 96% of the speed of light c. Calculate the distance the beam travels before its flux is reduced to half its initial flux. (1994) (Ans: 1.03 km)

(Hints: $d = V \times T_{1/2} = V \times \frac{T'_{1/2}}{\sqrt{1-\frac{V^2}{c^2}}} = (0.96c) \times \frac{2\times10^{-6}}{\sqrt{1-\frac{(0.96c)^2}{c^2}}}$

$= (0.96c) \times (3.57 \times 10^{-6} s) = 1.03km$)

20. By what factor the clock set at the frame moving with velocity 0.8c with respect to the rest frame will appear slower if noticed from the rest frame?
21. Two light sources A and B situated at 10m apart flash at an interval of 10^{-9}s. At what interval will an observer going at a speed of 0.9c in a direction from A to B parallel to the line joining the two sources will appear to him to flash first?
22. Two coordinate systems (x, y) and $(x^/, y^/)$ with a common origin admit the following transformation

$$x^/ = x\, cos\theta + y\, sin\theta$$
$$y^/ = -x\, sin\theta + y\, cos\theta$$

where θ is the angle between the x- axis and $x^/$- axis. Take ict for y and $ict^/$ for $y^/$, c being the velocity of light in vacuum. Now find a suitable value for θ so as to obtain the following transformation.

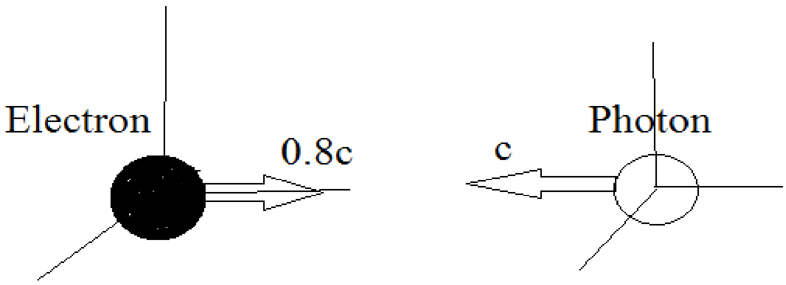
$$x^/ = \frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}} \qquad t^/ = \frac{t - \frac{V}{c^2}x}{\sqrt{1-\frac{V^2}{c^2}}}$$

23. How many times will the half life of an unstable particle increase, if the particle moves with a velocity of 0.99c? (1998) (Ans: 7.09 times)

(Hints: $\frac{T_{1/2}}{T'_{1/2}} = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.99c)^2}{c^2}}} = 7.09$, where $T_{1/2}, T'_{1/2} \rightarrow$ Half life times of the particle w.r. to laboratory frame and its own frame respectively.)

24. A particle with a mean proper life of 1 microsecond moves through the laboratory at a speed of 2.7×10^8m/s, what will be its life time as measured by an observer in the laboratory?
25. A rod has a length of 1 metre. When the rod is in a satellite moving with respect to Earth at a speed 0.99c, what is its length as determined by an observer in the satellite?
26. The rest radius of Earth is 6400 km and its orbital speed is 30 km/s. By how much would the Earth's diameter appear to be shortened to an observer on the Sun due the Earth's orbital motion?

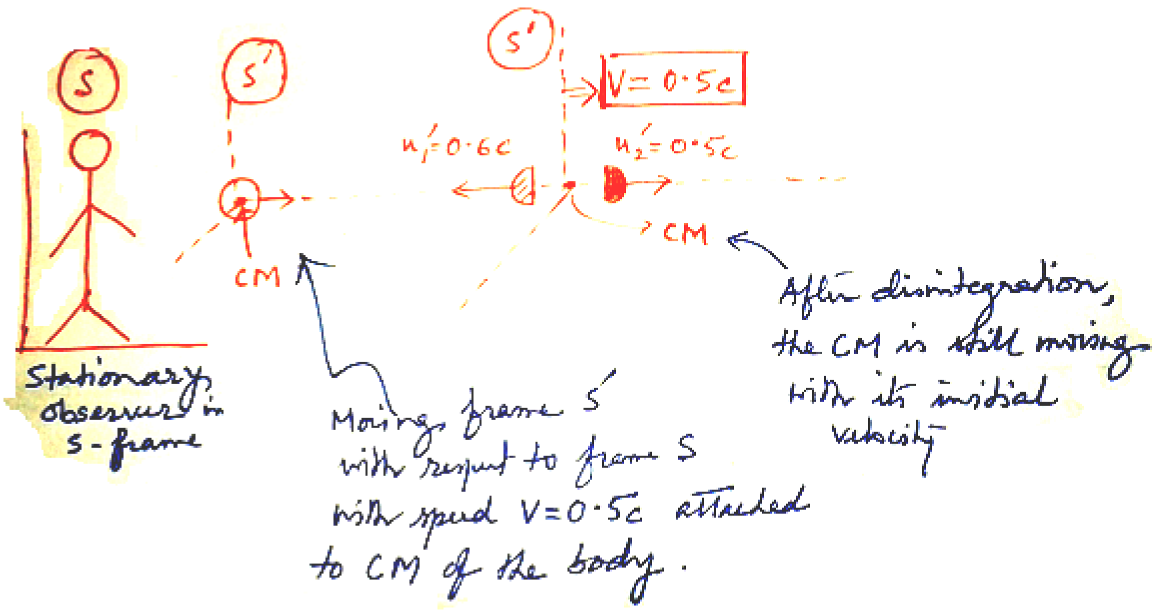
27. An electron is moving with a speed of 0.8c in a direction opposite to that of a moving photon. Calculate the relative velocity of the photon with respect to the electron.



(Hint: $u = \dfrac{u' + V}{1 + \dfrac{u'V}{c^2}} = \dfrac{c + 0.8c}{1 + \dfrac{c \times 0.8c}{c^2}} = c$)

28. Two elementary particles are approaching each other at speed 0.5c. Find the relative speed of one particle as seen from the rest frame of other.

29. A body moving at 0.5c with respect to an observer disintegrates into two fragments that move in opposite directions relative to their CM along the same line of motion as the original body. One fragment has a velocity of 0.6c in backward direction relative to the CM and the other has a velocity of 0.5c in forward direction. What velocities will the observer find for the fragments? (Ans: -0.129c, 0.8c)

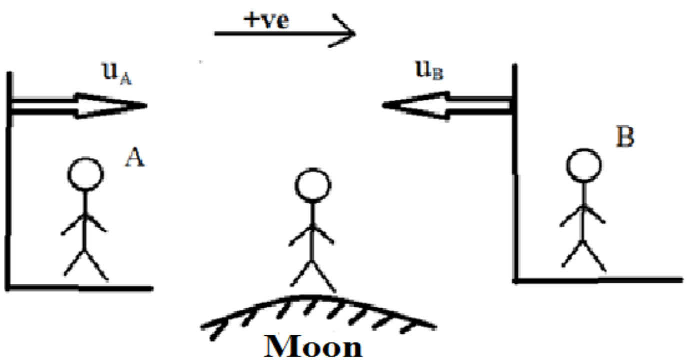


(Hints: If u_1' is the velocity of a fragment moving in opposite direction in the CM frame, then the velocity of that fragment w. r. to a stationary observer the velocity of the fragment

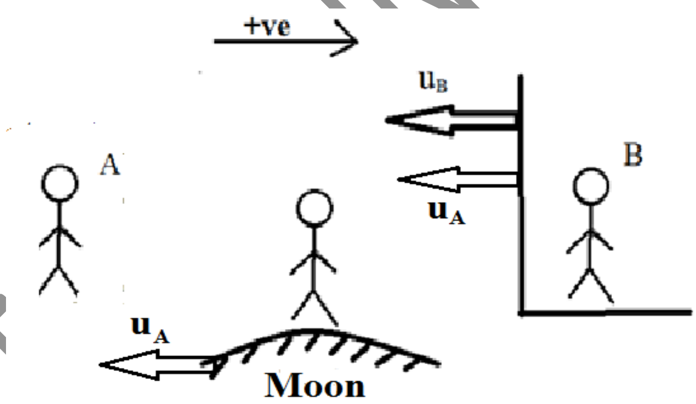
$$u_1 = \frac{u_1' + V}{1 + \dfrac{u_1'V}{c^2}} = \frac{(-0.6c) + (0.5c)}{1 + \dfrac{(-0.6c)(0.5c)}{c^2}} = -\frac{1}{7}c = -0.129c \text{ and}$$

$$u_2 = \frac{u_2' + V}{1 + \frac{u_2' V}{c^2}} = \frac{(0.5c) + (0.5c)}{1 + \frac{(0.5c)(0.5c)}{c^2}} = 0.8c\,)$$

30. A man on the moon sees two aircrafts A and B coming towards him from opposite directions at the respective speeds of 0.800c and 0.900c.
    a) What does a man on A measure for the speed with which he is approaching the moon?
    b) For the speed with which he is approaching B?



(Hints: a) vel. of the moon approaching A = $-u_A$=0.800c



From the point of view of A

b) vel. of B approaching A

$$= \frac{(-u_B) + (-V)}{1 + \frac{(-u_B)(-V)}{c^2}} = \frac{(-0.900c) + (-0.800c)}{1 + \frac{(-0.900c)(-0.800c)}{c^2}} = -0.988c$$

Mass Variation, Mass –Energy Equivalence

1. Derive the formula for variation of mass with velocity.
2. Derive the expression for mass energy equivalence.
3. Discuss the applications of mass energy equivalence.
   Or, Give two examples where mass-energy equivalence can be observed.

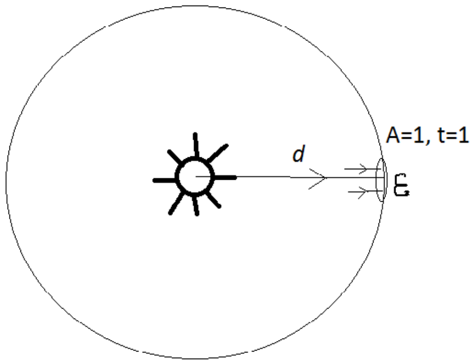
4. Show that relativistic KE of a moving particle is c^2 times the apparent increase in mass of the particle, where c is the free space speed of light.
   (Hints: $KE = \Delta m \times c^2$)
5. Use the transformation equation involving rest mass and moving mass to derive an expression for the total mass-energy of a moving body.
6. State the mass energy equivalence relation. Show that for small velocity $\left(\frac{v}{c} \ll 1\right)$, it yields the classical expressions for KE of a particle.
7. Show that a particle with zero rest mass must travel at the speed of light in vacuum.
8. Calculate the velocity of an electron moving with KE of 1 MeV, given that the rest mass of electron is 9.1×10^{-31}kg. What is the moving mass of the electron?
9. An electron of mass 9.1×10^{-31}kg moves with a speed of $0.9c$, where c is the free space speed of light. Calculate the relativistic KE and show that the value is greater than what is obtain from classical calculation.
10. The total energy of a moving meson is exactly thrice its rest energy. Find the speed of the meson.
11. The relativistic mass of a proton exceeds its rest mass by 1%. Calculate its speed of its rest mass is 1.67×10^{-27}kg.
12. At what velocity the mass of the particle becomes twice its rest mass?
13. A moving electron has energy of 0.50 MeV, what will be its corresponding mass?
14. Find the speed of a 0.1 MeV electron according to classical and relativistic mechanics.
15. Find the energy equivalent to the rest mass of the electrons and to the rest mass of proton?
16. Prove that $a = \frac{F}{m_0}\left(1-\frac{u^2}{c^2}\right)^{3/2}$.

    (Hints: $F = \frac{dP}{dt} = \frac{d(mu)}{dt} = \frac{dm}{dt}u + m\frac{du}{dt} = \frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\right)\times u + \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\times a =$..........)
17. Prove that $\frac{1}{2}mu^2$, where $m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$ **doers not** equal to the KE of a particle moving at a relativistic speed.
18. Solar energy reaches the Earth at the rate of about 1.4 kW per square metre of surface perpendicular to the direction of the Sun rays. By how much does the mass of the Sun decrease per second owing to this energy loss? The mean radius of earth's orbit is 1.5×10^{11}m.
    (Hints: $m = \frac{E}{c^2} = \frac{4\pi d^2 \varepsilon}{c^2}$

$$= \frac{4 \times 3.14 \times \left(1.5 \times 10^{11} m\right)^2 \times \left(1.4 \times 10^3 J/sm^2\right)}{\left(3 \times 10^8 m/s\right)^2}$$

$$= 4.4 \times 10^9 kg/s \text{)}$$

(Total mass of the Sun is 2.0×10^{30}kg)



19. An electron and a photon both have momenta of 2.000 MeV/c. Find the total energy of each.
    (Hints: for the electron
    $E_e = \sqrt{p^2c^2 + \left(m_0c^2\right)^2} = \sqrt{\left(2.000 MeV/c\right)^2 \times c^2 + \left(0.511 MeV\right)^2} = 2.064 MeV$
    For the photon $E_\gamma = pc = \left(2.000 MeV/c\right) \times c = 2.000 MeV$)
20. The Bevatron- a proton accelerator gives proton a KE of $10^{-2} erg$. By what factor is the mass of such protons increased? Rest mass $m_0 = 1.67 \times 10^{-24} g$.
    (Hints: $\frac{m}{m_0} = \frac{E\left(= K + m_0c^2\right)}{m_0c^2} = 1 + \frac{K}{m_0c^2} = 7.68$)
21. Dynamite liberates about 5.4×10^6J/kg when explodes. What fraction of its total energy content is this?
    (Hints: Apply $E = mc^2$, Energy obtained from 1 kg mass=$1kg \times \left(3 \times 10^8 m/s\right)^2$ $= 9 \times 10^{16} J$
    Fraction of liberated energy to total energy $= \frac{5.4 \times 10^6}{9 \times 10^{16}}$)
22. Find the speed and momentum (in GeV/c) of a proton whose total energy is 3.500 GeV. (1 GeV= 10^9 eV, Rest mass of proton=0.938 GeV/c^2)
    (Ans: v=0.963c, p=3.352 GeV/c)
    (Hints: $E = mc^2 = \left(\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}\right) c^2 = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$
    $E = K + m_0c^2 \qquad p^2c^2 = K^2 + 2Km_0c^2$)

23. Find the KE of an electron, its speed and its mass at the end of the acceleration in a potential field of 10^4 Volts.
(Hints: KE= $1eV = 1.6\times10^{-19}C\times10^4V = 1.6\times10^{-15}J$,
$m = \frac{KE}{c^2} + m_0 = 9.3\times10^{-31}kg$, $\quad \frac{m}{m_0} = \left(1-\frac{u^2}{c^2}\right)^{-1/2} \Rightarrow \frac{u}{c} = \left[1-\left(\frac{m_0}{m}\right)^2\right]^{1/2} = 0.195$)

24. A particle has a KE of 62 MeV and a momentum of 335 MeV/c. Find its rest mass (MeV/c^2) and speed (as a fraction of c). (Ans: 812 MeV/c^2, 0.37c)
(Hint: Apply $\quad k^2 + 2km_0c^2 = p^2c^2$

$$m_0 = \frac{p^2c^2 - k^2}{2kc^2} = \frac{(335\ \mathrm{MeV/c})^2c^2 - (62\ \mathrm{MeV})^2}{2\times 62\ \mathrm{MeV}\times c^2} = 812\ MeV/c^2$$

Now, put the value of m_0 in the following eqn to calculate speed.

$$p = mv = \frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow v = \left(1+\frac{m_0c^2}{p^2}\right)^{-\frac{1}{2}}c = 0.37c)$$

25. What is the energy contained in 1g of coal? How does this compare with the 7000 calories of heat delivered by burning 1g of coal?
(Ans: Rest mass energy of 1g of coal is 3.1×10^9 as much as the chemical energy liberated in the form of heat during the burning of 1g of coal.)

26. An electron whose speed relative to an observer in a laboratory is 0.800c is also being observed by an observer moving in the same direction as the electron at a speed of 0.500c relative to the laboratory. What is the KE (in MeV) of the electron to each observer? (Ans: 0.341MeV, 0.077MeV)
(Hints: For the 1st observer at rest in the lab frame

$$KE = \Delta m\times c^2 = (m-m_0)\times c^2 = \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}-1\right)m_0c^2$$

For the 2nd observer in motion in the lab frame

$$KE = \Delta m\times c^2 = (m'-m_0)\times c^2 = \left(\frac{1}{\sqrt{1-\frac{u'^2}{c^2}}}-1\right)m_0c^2$$, Here $u' = \frac{u-V}{1-\frac{uV}{c^2}}$ =0.500c is

the speed of the electron with w. r. to the moving observer with the speed V.)

27. Verify that $\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 1+\frac{KE}{m_0c^2}$.

(Hint: Apply $E = mc^2 = KE + m_0c^2 \Rightarrow \frac{m}{m_0} = \frac{KE}{m_0c^2} + 1, \quad m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$)

28. Verify that $\dfrac{1}{\sqrt{1-\dfrac{u^2}{c^2}}} = \sqrt{1+\dfrac{p^2}{m_0^2 c^2}}$

(Hint: $E^2 = (mc^2)^2 = p^2c^2 + m_0^2c^4 \Rightarrow \left(\frac{m}{m_0}\right)^2 = \frac{p^2}{m_0^2c^2} + 1, \quad m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$)

29. For what value of $\dfrac{u}{c}(=\beta)$ will the relativistic mass of a particle exceed its rest mass by a given factor '*f*'?

(Hint: $f = \dfrac{\Delta m}{m} = \dfrac{m-m_0}{m} = 1-\left(1-\dfrac{u^2}{c^2}\right)^{-1/2} \Rightarrow \dfrac{u}{c} = \dfrac{\sqrt{f(f+2)}}{(1+f)}$)

30. At what speed does the KE of a particle equal to its rest mass energy? (Ans: $\dfrac{\sqrt{3}}{2}c$)

(Hints: $KE = m_0c^2 \Rightarrow [m-m_0]c^2 = m_0c^2 \Rightarrow u = \dfrac{\sqrt{3}}{2}c$)

31. Find the momentum of an electron whose KE equals its rest mass energy of 511 KeV.

(Hints: $p^2c^2 = K^2 + 2Km_0c^2 = K^2 + 2K \times K = 3K^2$

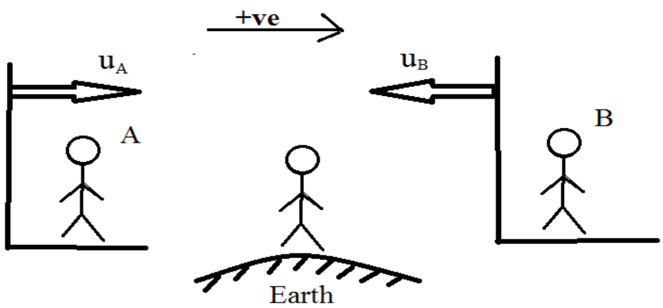
$$\Rightarrow p = \sqrt{\frac{3K^2}{c^2}} = \sqrt{3}\frac{K}{c} = \sqrt{3}\times(511KeV/c) = 885.1KeV/c)$$

32. Find the momentum of an electron whose speed is 0.600c.

(Hints: $p = mu = \dfrac{m_0}{\sqrt{1-\dfrac{u^2}{c^2}}}u = \dfrac{m_0c^2}{\sqrt{1-\dfrac{u^2}{c^2}}}\left(\dfrac{u}{c}\right)\dfrac{1}{c}$

$$= \frac{0.511MeV}{\sqrt{1-\dfrac{(0.600c)^2}{c^2}}}\left(\frac{0,600c}{c}\right)\frac{1}{c} = 0.383MeV/c)$$

33. Two twins of rest mass 60.0kg are headed towards each other in spacecraft whose speed relative to the earth are 0.800c. What mass does each twin find for the other?



(Hints: Speed of B heading towards A w.r.t. to A

$$u = \frac{u' + V}{1 + \frac{u'V}{c^2}} \Rightarrow u_B = \frac{(-0.800c) + (-0.800c)}{1 + \frac{(-0.800c)(-0.800c)}{c^2}} = -0.976c$$

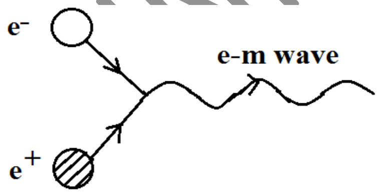
(-ve sign indicates that B is moving in the opposite direction of the motion of A)

$m_B = \dfrac{m_0}{\sqrt{1 - \frac{u_B^2}{c^2}}} = \dfrac{60.0kg}{\sqrt{1 - \frac{(0.976c)^2}{c^2}}} = 275.5kg$, etc.)

34. Calculate the velocity of an electron moving with KE of 1 MeV, given that rest mass of electron is $9.1 \times 10^{-31} kg$. What is the moving mass of the electron?

(Hints: $E = mc^2 = K + m_0c^2$, $m = \dfrac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \Rightarrow \dfrac{u}{c} = \left[1 - \left(\dfrac{E}{m_0c^2}\right)^2\right]^{1/2} \Rightarrow u = 0.94c$)

35. A positron collides head-on with an electron and both are annihilated. Each particle had a KE of 1MeV. Find the wavelength of the resulting photon. (Ans: 0.0041 A^0)



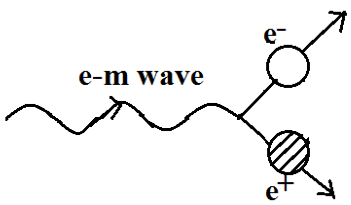
Pair annihilation

(Hints: From energy conservation principle

$E_\gamma = E_{e^-} + E_{e^+} \Rightarrow h\nu = \left(K + m_0c^2\right)_{e^-} + \left(K + m_0c^2\right)_{e^+}$, $\lambda = \dfrac{c}{\nu}$

$\left(m_0c^2\right)_{e^-} = \left(m_0c^2\right)_{e^+} = 9.1 \times 10^{-31} kg \times 3 \times 10^8 m/s = 0.511 MeV$)

36. Suppose an electron and a positron at rest come together and annihilate each other producing two photons of equal energy. Find the energy of each photon.
37. Find the minimum energy of a gamma ray photon in MeV which can cause $e^- - e^+$ pair production. (GU) (Ans 1-042 MeV)



Pair production

(Hints: $E_\gamma = E_{e^-} + E_{e^+} \Rightarrow h\nu = \left(m_0c^2\right)_{e^-} + \left(m_0c^2\right)_{e^+}$, etc.)

Minkowski Diagram

1. Write a short note on **Minkowski Diagram**.
2. Define a **world line**.
3. Take a Minkowski Diagram with two axes x and ω (ω=ct) perpendicular to each other. Show that in this diagram, the **world line of light** is a straight line making a 45^0 angle with either axis whereas the tangent to the world line of a material particle makes an angle less than 45^0 with the ω axis.
4. Show how one can arrive at
   (i) the relativity of simultaneity and
   (ii) the length contraction from space-time diagram.
5. Explain with examples the meaning of 'time-like' and 'space-like' coordinates.
6. What are time-like, space-like and light-like intervals?

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