edfabiklzxcv rtyuio **Class Notes on A, B, C of Tensor Algebra for BSc (M)** asdfgh Arup Bharali **Bhattadev** University Pathsala

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# PART-I

# **1. Introduction**

The tensor analysis was first formulated by **G. Ricci** in 1887 and subsequently developed by **Levi-Civita** and others. In many physical problems, vector analysis is **not** sufficiently general to express the relations among the physical quantities involved, but the tensor analysis helps us in generalising the formulation of these relationships.

The basic advantage of using tensor is that the results expressed in terms of tensors remain <u>invariant</u> under the transformation of coordinates from one coordinate system to another. It is a fundamental fact that all the physical laws of nature are independent of the choice of the coordinate system used to state them mathematically (echoing the Einstein's Principle of Relativity). So Einstein applied tensor as the natural tool for the formulation of General Theory of Relativity.

Tensors are also applied for the study of Dynamics, Hydrodynamics, Electrical as well as Optical properties of anisotropic solids, etc.

#### Illustrations

1. As an introductory illustration, we cosider the flow of electric current. The usual vector form of Ohm's Law is

$$\vec{J} = \sigma \vec{E}$$
 .....(1)

Where  $\vec{J}$  is the current density and  $\vec{E}$  is the electric field intensity and both are vectors. If the medium is **isotropic**, the **electrical conductivity**  $\sigma$  is a **scalar** and

both  $\vec{J}$  and  $\vec{E}$  are parallel, meaning is that the current is established inside the medium in the direction of applied electric field and hence



Hence

$$J_i = \sigma E_i \qquad \text{i=1, 2,3}$$

in case of 3D Cartesian coordinate system  $(X_1, X_2, X_3)$ .

However if the medium is **anisotropic**, as in many crystals, the current density  $J_1$  along  $X_1$ -direction may depend on the electric fields along  $X_2$  and  $X_3$ -directions as well as along  $X_1$ -direction. So,

$$J_1 \propto E_1, E_2, E_3$$

and we assume the following **linear relationship** to replace the equation in (2a)

Similarly

Combining all the above three equations (3a), (3b) and (3c) in 3D cases

$$J_{i} = \sum_{j=1}^{3} \sigma_{ij} E_{j} = \sigma_{ij} E_{j} \qquad i = 1, 2, 3.$$

#### (by using Einstein's summation convention)

The first suffix of  $\sigma_{ij}$  indicates the direction of J and the second suffix for that of E

Thus the scalar electrical conductivity  $\sigma$  has been given way to a set of nine (09) elements  $\sigma_{ij}$  (*i*, *j* = 1,2,3) for an ordinary 3D space.

$$\sigma_{ij} = (\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{22}, \sigma_{23}, \sigma_{31}, \sigma_{32}, \sigma_{33})$$
 is a wrong way of writing  $\sigma$ 

And the correct way of representative of  $\sigma_{ij}$  is as follows.

 $\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \text{ and } \sigma_{ij} \text{ is the components of the electrical conductivity}$ 

tensor.

• The array of nine elements of  $\sigma_{ij}$  forms a tensor. But this does not mean that any square array of numbers or function forms a tensor. The essential condition is that the components transform according to a **specific rule**.

2. If the medium is **isotropic dielectric**, the relation between the polarisation vector  $\vec{P}$  and electric field  $\vec{E}$  is given by

$$\vec{P} = \varepsilon \vec{E}$$

where  $\varepsilon$  is the scalar dielectric constant. The x-component of  $\vec{P}$  is given by  $P_1 = \varepsilon E_1$ ,

Similarly,

$$P_2 = \varepsilon E_2$$
$$P_3 = \varepsilon E_3$$

in 3D XYZ system. The general form is

$$P_i = \varepsilon E_i$$
 i=1, 2, 3

However, if the medium is **anisotropic dielectric**, the x-component of  $\vec{P}$  may also depend on y and z components of  $\vec{E}$  as long as x-component of  $\vec{E}$ , i.e.

$$P_1 \propto E_1, E_2, E_3$$

Assuming the linear relation-

$$P_1 = \varepsilon_{11}E_1 + \varepsilon_{12}E_2 + \varepsilon_{13}E_3 = \sum_{j=1}^{5} \varepsilon_{1j}E_j$$

Similarly

$$P_{2} = \varepsilon_{21}E_{1} + \varepsilon_{22}E_{2} + \varepsilon_{23}E_{3} = \sum_{j=1}^{3} \varepsilon_{2j}E_{j}$$
$$P_{3} = \varepsilon_{31}E_{1} + \varepsilon_{32}E_{2} + \varepsilon_{33}E_{3} = \sum_{j=1}^{3} \varepsilon_{3j}E_{j}$$

In general, for an ordinary 3D space

$$P_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j = \varepsilon_{ij} E_j, \ (i, j = 1, 2, 3)$$

where  $\mathcal{E}_{ii}$  is the dielectric tensor with nine elements and can be written as

 $\boldsymbol{\varepsilon}_{ij} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$ 

# 2. General Rules

- 1. Tensors have no special notation unlike vectors  $(\vec{A}, \vec{F}, \vec{v}, ....)$ .
- 2. Tensors are expressed in terms of their components, viz-  $A_i, B_{ij}, C_{ij}^k, \dots$

There are two ways of specification of tensor components, one by **superscripts** and the other by **subscripts**. The suffices i, j in  $A_j^i$  are called **superscript** and **subscript** respectively. The superscript does not mean the power index.

- **3.** Except zeroth rank tensors (scalars), first rank tensors (vectors) and second rank tensors ( $\sigma_{ii}, \varepsilon_{ii}, ...$ ), higher rank tensors can not be **visualised**.
- 4. The rank of a tensor is equal to the number of suffices or indices attached to it. E.g.  $C_{ii}^k$  is a component of  $3^{rd}$  rank tensor.

#### **Definition of RANK of a tensor:**

Rank of tensor gives the **number of mode of changes** of a physical quantity when passing from one coordinate system to another one which is **in rotation relative** to the first.

#### Zeroth rank tensors

Scalars or invariants do not get changed when the axes are rotated and so mode of change for scalars is zero. So, they are zeroth rank tensors.

Scalars	$a(x^{1}, x^{2}, x^{3}, \dots, x^{N}) \xrightarrow{R(\theta)} \overline{a}(x^{1}, x^{2}, x^{3}, \dots, x^{N}) = a$
Dot product	$m = \overrightarrow{A}.\overrightarrow{B} \xrightarrow{R(\theta)} \overrightarrow{m} = \overrightarrow{\overrightarrow{A}}.\overrightarrow{\overrightarrow{B}} = \overrightarrow{A}.\overrightarrow{B} = m$
Scalar function	$\phi(x^1, x^2, x^3, \dots, x^N) \xrightarrow{R(\theta)} \overline{\phi}(x^1, x^2, x^3, \dots, x^N) = \phi$

A zero rank tensor is simply defined by a single number regardless of any coordinate system and hence they are invariants under coordinate transformation.

# First rank tensors

The first rank tensors need for their complete identification a **number**, representing the magnitude and a **direction** representing their geometric orientation within their space.

#### **Examples of tensors of different rank**

Rank	Examples
Zero	Temperature, energy, mass, volume, density, dot product of two polar or axial vectors, etc.
One	Displacement, velocity, acceleration, force, electric field, etc. (vectors)
Two	Strain, stress, inertia, electrical conductivity, permittivity, Kronecker Delta, etc.

Three	Piezoelectric moduli, Levi-Civita, etc.
Four	Elastic/Stiffness tensor,

Tensors of higher rank are relatively rare in science and engineering

**5.** The number of components of a tensor of a particular rank in any dimensional space can be calculated using the following formula.

#### Number of components=(DIMENSION)<sup>RANK</sup>

Thus, a tensor of rank R in D dimensional space has D<sup>R</sup> number of components.

**6.** A) If all the components of a tensor in one coordinate system vanish at a point P, they vanish at that point in every coordinate system.

B) In the same way, if the components of two tensors are equal in one coordinate system, they are equal in all coordinate system.

- 7. The transformation properties of an object determines whether or not it is a tensor.
- 8. A repeated index in a term indicates summation, by *Einstein's summation convention* and it is also called a **dummy** or **umbral index**. A repeated index can be substituted by another new index.

An index occurring only once in a given term is called a free index.

For example, in the term  $a_{ij}x^j$ , the index j is the dummy (repeated) index and index i is the free index. By Einstein's summation convention,  $a_{ij}x^j = \sum_{j=1}^N a_{ij}x^j \left(=\sum_{m=1}^N a_{im}x^m\right).$ 

**Problem:** Mention whether tensors  $a_i^{\mu} x^i$  and  $a_i^{\sigma} x^i$  are same or not. (2018)

Hint: Here i index is the repeated or dummy index, while  $\mu$  and  $\sigma$  are the free indices. A repeated index may be substituted with some other new index, but if the free index is substituted with a new one, the tensor gets changed to another one.

# **3.** Transformation rules for contravariant tensors

**3a. Transformation properties/rule for the components of radius vector under rotation of coordinate axes in 2D plane** 



Transformation rules for the components of the radius vector r are

$$\begin{aligned} \bar{x}^{-1} &= x^{1}\cos\theta + x^{2}\sin\theta = x^{1}\cos\theta + x^{2}\cos\left(\frac{\pi}{2} - \theta\right) = x^{1}\cos(\bar{X}^{1}, X^{1}) + x^{2}\cos(\bar{X}^{1}, X^{2}) \\ &= a_{11}x^{1} + a_{12}x^{2} = \sum_{i=1}^{2} a_{1i}x^{i} \end{aligned}$$

(From the figure

$$\overline{x}^{-1} = ob = oa + ab = cg + gp = de + gp = x^2 \sin \theta + x^1 \cos \theta$$

and

$$\begin{aligned} \overline{x}^{2} &= -x^{1}\sin\theta + x^{2}\cos\theta = x^{1}\cos\left(\frac{\pi}{2} + \theta\right) + x^{2}\cos\theta = x^{1}\cos(\bar{X}^{2}, X^{1}) + x^{2}\cos(\bar{X}^{2}, X^{2}) \\ &= a_{21}x^{1} + a_{22}x^{2} = \sum_{i=1}^{2} a_{2i}x^{i} \end{aligned}$$

(From the figure

$$x^{-2} = oc = od - cd = ae - eg = x^2 \cos \theta - x^1 \sin \theta$$

In compact form

$$\bar{x}^{p} = \sum_{i=1}^{2} a_{pi} x^{i} \text{ or } \bar{x}^{p} = a_{pi} x^{i}, \qquad (i, p = 1, 2)$$

where  $a_{pi} = \cos\left(\overline{X}^{p}, X^{i}\right) = \text{cosine of the angle between } \overline{X}^{p}$  and  $X^{i}$  axes.

Differentiating the above equation w.r. to  $x^k$ , we get

$$\frac{\partial \overline{x}^{p}}{\partial x^{k}} = \frac{\partial}{\partial x^{k}} \left( a_{pi} x^{i} \right) = a_{pi} \frac{\partial x^{i}}{\partial x^{k}} = a_{pi} \delta_{k}^{i} = a_{pk},$$

where  $a_{pk}$  is called the **transformation factor or cofactor** and  $\delta_k^i$  is the **Kronecker Delta**, which property is

$$\delta_j^i = \frac{\partial x^i}{\partial x^j} = 1$$
, when  $i = j$   
= 0, when  $i \neq j$ 

Hence,

$$\overline{x}^p = a_{pi} x^i \longrightarrow \overline{x}^p = \left(\frac{\partial \overline{x}^p}{\partial x^i}\right) x^i.$$

The above transformation rule is used to define any component of the radius vector  $\vec{r}$  in a new 2D coordinate system  $(\vec{x}^{v}, p=1, 2 \text{ system})$  in terms of the components in the old 2D coordinate system  $(x^{i}, i=1, 2 \text{ system})$ , if and only if the new system gets a rotation relative to the old one.

The above transformation rule can be extrapolated to coordinate systems of N dimensions. If  $x^i$  be the ith component of radius vector  $\vec{r}$  in  $(x^1, x^2, x^3, \dots, x^N)$  coordinate system of N dimensions, then the pth component  $\overline{x}^p$  of radius vector  $\vec{r}$  in  $(\overline{x}^1, \overline{x}^2, \overline{x}^3, \dots, \overline{x}^N)$  coordinate system of N dimensions is given by

$$\overline{x}^{p} = \sum_{i=1}^{N} \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) x^{i} = \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) x^{i}, \qquad i, p = 1, 2, 3, \dots, N$$

if and only if, the new coordinate system is a rotated system relative to the old system.

• Any set of quantities  $A^i$  which transforms like the components of radius vector as given by the above equation is defined as the **contravariant vector** or **contravariant tensor of rank one**. Therefore, the transformation property for a first rank contravariant tensor  $A^i$  is

$$\overline{A}^{p} = \sum_{i=1}^{N} \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) A^{i} = \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) A^{i}, \qquad i, p = 1, 2, 3, \dots, N$$

#### Definition of Contravariant tensor of rank one

If N quantities  $A^1, A^2, A^3, \dots, A^N$  in the  $x^i$  ( $i = 1, 2, 3, \dots, N$ ) coordinate system of N dimensions are related to N quantities  $\overline{A}^1, \overline{A}^2, \overline{A}^3, \dots, \overline{A}^N$  in the  $\overline{x}^p$  ( $p = 1, 2, 3, \dots, N$ )-coordinate system of N dimensions by means of the following equation

$$\overline{A}^{p} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{1}}\right) A^{1} + \left(\frac{\partial \overline{x}^{p}}{\partial x^{2}}\right) A^{2} + \dots + \left(\frac{\partial \overline{x}^{p}}{\partial x^{N}}\right) A^{N}$$
  
or 
$$\overline{A}^{p} = \sum_{i=1}^{N} \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) A^{i} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) A^{i} ,$$

then the N quantities  $A^1, A^2, A^3, \dots, A^N$  are called the components of a contravariant tensor of rank one (vector) in the  $x^i$ -coordinate system, and  $\overline{A}^1, \overline{A}^2, \overline{A}^3, \dots, \overline{A}^N$  are the components of the same contravariant tensor of rank one (vector) in the  $\overline{x}^p$ -coordinate system.

#### Displacement as a 1st rank contravariant tensor

Since  

$$x^{i} \xrightarrow{Coordinate Transformation} \xrightarrow{\pi^{p}} \overline{x}^{p}$$

$$dx^{i} \xrightarrow{Coordinate Transformation} \xrightarrow{dx^{p}} d\overline{x}^{p}$$

$$\vdots$$

$$\overline{x}^{p} = \overline{x}^{p} \left(x^{1}, x^{2}, ..., x^{N}\right), \qquad p = 1, 2, ..., N$$

$$\therefore \quad d\overline{x}^{p} = \frac{\partial \overline{x}^{p}}{\partial x^{1}} dx^{1} + \frac{\partial \overline{x}^{p}}{\partial x^{2}} dx^{2} + ..., + \frac{\partial \overline{x}^{p}}{\partial x^{N}} dx^{N} = \sum_{i=1}^{N} \frac{\partial \overline{x}^{p}}{\partial x^{i}} dx^{i} \quad (By chain rule)$$

$$\Rightarrow d\overline{x}^{p} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) dx^{i} \qquad (By Einstein's summation convention)$$

The above transformation rule for the components of a displacement vector respects that of for 1<sup>st</sup> rank contravariant tensor, so displacement is a 1<sup>st</sup> rank contravariant tensor.

#### • Velocity as a 1st rank contravariant tensor

$$x^{i} \xrightarrow{Coordinate Transformation} \overline{x}^{p}$$

$$V^{i} = \frac{dx^{i}}{dt} \xrightarrow{Coordinate Transformation} \overline{V}^{p} = \frac{d\overline{x}^{p}}{dt}$$

$$\overrightarrow{V}^{p} = \frac{d\overline{x}^{p}}{dt} = \frac{\partial\overline{x}^{p}}{\partial x^{1}} \frac{dx^{1}}{dt} + \frac{\partial\overline{x}^{p}}{\partial x^{2}} \frac{dx^{2}}{dt} + \dots + \frac{\partial\overline{x}^{p}}{\partial x^{N}} \frac{dx^{N}}{dt} \quad \text{(By chain rule)}$$

$$= \sum_{i=1}^{N} \frac{\partial\overline{x}^{p}}{\partial x^{i}} \frac{dx^{i}}{dt} = \left(\frac{\partial\overline{x}^{p}}{\partial x^{i}}\right) \frac{dx^{i}}{dt} \quad \text{(By Einstein's summation convention)}$$

$$\Rightarrow \overline{V}^{p} = \left(\frac{\partial\overline{x}^{p}}{\partial x^{i}}\right) V^{i}$$

The above transformation rule for the components of a velocity vector respects that of for 1<sup>st</sup> rank contravariant tensor, so velocity is a 1<sup>st</sup> rank contravariant tensor.

• Acceleration as a 1st rank contravariant tensor

$$x^{i} \xrightarrow{\text{Coordinate Transformation}} \overline{x}^{p}$$

$$a^{i} = \frac{dV^{i}}{dt} \xrightarrow{\text{Coordinate Transformation}} \overline{a}^{p} = \frac{d\overline{V}^{p}}{dt}$$

$$\therefore \quad \overline{a}^{p} = \frac{d\overline{V}^{p}}{dt} = \frac{d}{dt} \left[ \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) V^{i} \right]$$

(Applying transformation rule for 1<sup>st</sup> rank contravariant tensor)

$$= \left[\frac{d}{dt}\left(\frac{\partial \overline{x}^{P}}{\partial x^{i}}\right)\right] V^{i} + \left(\frac{\partial \overline{x}^{P}}{\partial x^{i}}\right) \frac{dV^{i}}{dt} = 0 \times V^{i} + \left(\frac{\partial \overline{x}^{P}}{\partial x^{i}}\right) a^{i}$$
$$\overline{a}^{P} = \left(\frac{\partial \overline{x}^{P}}{\partial x^{i}}\right) a^{i}$$

• The above transformation rule/law/property for 1<sup>st</sup> rank contravariant tensor can be extrapolated to higher rank contravariant tensors.

#### **3b. Transformation rule for Contravariant tensor of rank two**

#### (Definition of Contravariant tensor of rank two)

If N<sup>2</sup> quantities  $A^{ij}$  (i, j = 1, 2, 3, ..., N) in the  $x^i$  (i = 1, 2, 3, ..., N)-coordinate system of N dimensions are related to N<sup>2</sup> quantities  $\overline{A}^{pq}$  (p, q = 1, 2, 3, ..., N) in the  $\overline{x}^{p}$  (p = 1, 2, 3, ..., N)-coordinate system of N dimensions by means of the following equation

$$\overline{A}^{pq} = \sum_{i,j=1}^{N} \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) \left( \frac{\partial \overline{x}^{q}}{\partial x^{j}} \right) A^{ij} = \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) \left( \frac{\partial \overline{x}^{q}}{\partial x^{j}} \right) A^{ij},$$

then the N<sup>2</sup> quantities  $A^{ij}$  are called the components of a contravariant tensor of rank two in the *x*-coordinate system, and  $\overline{A}^{pq}$  are the components of the same contravariant tensor of rank two in the  $\overline{x}$ -coordinate system.

#### 3c. Transformation Rule for Contravariant tensor of rank three

$$\overline{A}^{pqr} = \sum_{i,j,k=1}^{N} \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) \left( \frac{\partial \overline{x}^{q}}{\partial x^{j}} \right) \left( \frac{\partial \overline{x}^{r}}{\partial x^{k}} \right) A^{ijk} = \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) \left( \frac{\partial \overline{x}^{q}}{\partial x^{j}} \right) \left( \frac{\partial \overline{x}^{r}}{\partial x^{k}} \right) A^{ijk}, \qquad p,q,r = 1,2,3,\dots,N$$

# 4. Transformation rules for covariant tensors

# 4a. Transformation rules/properties for the components of gradient of a scalar function under coordinate transformations

Gradient of a scalar function  $\phi = \phi(x, y, z)$  in 3D XYZ coordinate system is defined as

$$grad\phi = \vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \sum \hat{i}\frac{\partial\phi}{\partial x}$$

It is a vector quantity. Extrapolating the above expression from three to an N dimensional space, we can write down the gradient as

$$grad\phi = \frac{\partial \phi(x^i)}{\partial x^i}, \qquad i = 1, 2, 3, \dots, N$$

Now we make a coordinate transformation as follows.

$$x^{i} \xrightarrow{Coordinate Transformation} \overline{x}^{F}$$

And thus, the transformation of different components of  $grad\phi$  become

$$\frac{\partial \phi(x^i)}{\partial x^i} \xrightarrow{Coordinate Transformation} \frac{\partial \overline{\phi}(\overline{x}^p)}{\partial \overline{x}^p}$$

(in the old x-coordinate system) (in the new  $\bar{x}$ -coordinate system)

 $\frac{\overline{\phi}\left(\overline{x}^{p}\right)}{\partial \overline{x}^{p}} = \frac{\partial \phi}{\partial \overline{x}^{p}} \qquad (\text{since } \overline{\phi} = \phi \text{ is a scalar function})$ 

$$= \frac{\partial \phi}{\partial x^1} \frac{\partial x^1}{\partial \overline{x}^p} + \frac{\partial \phi}{\partial x^2} \frac{\partial x^2}{\partial \overline{x}^p} + \dots + \frac{\partial \phi}{\partial x^N} \frac{\partial \phi}{\partial \overline{x}^N}$$

(since  $\phi = \phi(x^1, x^2, x^3, \dots, x^N)$  and using chain rule)

$$=\sum_{i=1}^{N}\frac{\partial\phi}{\partial x^{i}}\frac{\partial x^{i}}{\partial \overline{x}^{p}}$$

Hence

$$\frac{\partial \overline{\phi}}{\partial \overline{x}^{p}} = \left(\frac{\partial x^{i}}{\partial \overline{x}^{p}}\right) \frac{\partial \phi}{\partial x^{i}}, \qquad i, p = 1, 2, 3, \dots, N$$

That is the transformation rule for the components of gradient of a scalar function  $\phi = \phi(x, y, z)$ 

Whenever the transformation rule for a quantity (tensor) resembles the transformation rule for the components of gradient of a scalar function  $\phi = \phi(x, y, z)$  than that quantity (tensor) is called a covariant tensor of rank one.

#### Definition of covariant tensor of rank one

If N quantities  $A_1, A_2, A_3, \dots, A_N$  in the  $x^i$  ( $i = 1, 2, 3, \dots, N$ )-coordinate system of N dimensions are related to N quantities  $\overline{A}_1, \overline{A}_2, \overline{A}_3, \dots, \overline{A}_N$  in the  $\overline{x}^p$  ( $p = 1, 2, 3, \dots, N$ )-coordinate system of N dimensions by means of the following equation

$$\overline{A}_{p} = \left(\frac{\partial x^{1}}{\partial \overline{x}^{p}}\right) A_{1} + \left(\frac{\partial x^{2}}{\partial \overline{x}^{p}}\right) A_{2} + \dots + \left(\frac{\partial x^{N}}{\partial \overline{x}^{p}}\right) A_{N}$$

or 
$$\overline{A}_p = \sum_{i=1}^N \left( \frac{\partial x^i}{\partial \overline{x}^p} \right) A_i = \left( \frac{\partial x^i}{\partial \overline{x}^p} \right) A_i$$
,

then the N quantities  $A_1, A_2, A_3, \dots, A_N$  are called the components of a covariant tensor of rank one (vector) in the  $x^i$ -coordinate system, and  $\overline{A}_1, \overline{A}_2, \overline{A}_3, \dots, \overline{A}_N$  are the components of the same covariant tensor of rank one (vector) in the  $\overline{x}^p$ -coordinate system.

# • The above transformation rule/law/property for 1<sup>st</sup> rank covariant tensor can be extrapolated to higher rank covariant tensors.

# 4b. Transformation rules/properties for 2<sup>nd</sup> rank covariant tensor

# (Definition of Covariant tensor of rank two)

If N<sup>2</sup> quantities  $A_{ij}$  (i, j = 1, 2, 3, ..., N) in the  $x^i$  (i = 1, 2, 3, ..., N)-coordinate system of N dimensions are related to N<sup>2</sup> quantities  $\overline{A}_{pq}$  (p,q = 1, 2, 3, ..., N) in the  $\overline{x}^p$  (p = 1, 2, 3, ..., N)-coordinate system of N dimensions by means of the following equation

$$\overline{A}_{pq} = \sum_{i,j=1}^{N} \left( \frac{\partial x^{i}}{\partial \overline{x}^{p}} \right) \left( \frac{\partial x^{j}}{\partial \overline{x}^{q}} \right) A_{ij} = \left( \frac{\partial x^{i}}{\partial \overline{x}^{p}} \right) \left( \frac{\partial x^{j}}{\partial \overline{x}^{q}} \right) A_{ij} ,$$

then the N<sup>2</sup> quantities  $A_{ij}$  are called the components of a covariant tensor of rank two in the x-coordinate system, and  $\overline{A}_{pq}$  are the components of the same covariant tensor of rank two in the x-coordinate system.

# 4c. Transformation Rule for Contravariant tensor of rank three

$$\overline{\mathbf{A}}_{pqr} = \sum_{i,j,k=1}^{N} \left( \frac{\partial x^{i}}{\partial \overline{x}^{r}} \right) \left( \frac{\partial x^{j}}{\partial \overline{x}^{q}} \right) \left( \frac{\partial x^{k}}{\partial \overline{x}^{r}} \right) \mathbf{A}_{ijk} = \left( \frac{\partial x^{i}}{\partial \overline{x}^{p}} \right) \left( \frac{\partial x^{j}}{\partial \overline{x}^{q}} \right) \left( \frac{\partial x^{k}}{\partial \overline{x}^{r}} \right) \mathbf{A}_{ijk},$$
$$p,q,r = 1, 2, 3, \dots, N$$

# 5. Contravariant and covariant tensors - In what sense they differ one from the other?

The difference between these two kinds of tensors is how they transform under a continuous change of coordinates. Comparing the transformation rules of covariant tensor with that of contravariant tensor, it is seen that they both define the transformed components as **linear combination** of the original (old) components (see the definition of contravariant and covariant tensors of rank one), but in the contravariant case the coefficients **Class Notes on A, B, C of Tensor Algebra for B.Sc. (M).....Arup Bharali** 

(transformation factor) are the partial derivatives of the new coordinates w.r.t. the old (i.e.  $\left(\frac{\partial \bar{x}^p}{\partial x^i}\right)$ , etc.), whereas in the covariant case, the coefficients are the partial derivatives of the

old coordinates w.r.t. the new (i.e.  $\left(\frac{\partial x^i}{\partial \overline{x}^p}\right)$ , etc.).

• In general, any given vector or tensor can be expressed in both contravariant and covariant form w.r.t. any given coordinate system. For example,



(a)  $P^1$  and  $P^2$  are contravariant components of  $\vec{P}$  in the  $(X^1, X^2)$  coordinate system. (e.g. displacement vector) The 'contra' components 'go with' the axis, i.e. the jth component of a

contravariant vector  $\vec{P}$  is the projection of  $\vec{P}$  onto the jth axis parrel to the other axis.

(b)  $P_1$  and  $P_2$  are the covariant components of  $\vec{P}$  in the  $(X^1, X^2)$  coordinate system. (e.g. gradient of a scalar field, which is a vector)

The 'co' components 'go against' the axes, i.e. the jth component of a covariant vector  $\vec{P}$  is the projection of  $\vec{P}$  into the jth axis perpendicular to that axis.

- (c) It may seem that the naming convention is backwards, because the "contra" components go with the axes, whereas the "co" components go against the axes, but historically these names were given **on the basis on the transformation laws** that apply to these two different interpretations.
- (d) Those terms contravariant or covariant really just signify two different conventions for interpreting the <u>components</u> of the object with respect
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to a given coordinate system, whereas the essential attributes of a vector or tensor are independent of the particular coordinate system in which we choose to express it.

- (e) For orthogonal coordinate system, contravariant and covariant components are identical. That can be seen by imagining that coordinate axes in the figure are perpendicular to each other.
- (f) The contravariant and covariant types of tensor are inked through the **metric tensor**.

#### 6. Finally......Definition of tensor:

A tensor is an array of mathematical or physical objects (usually numbers or functions) which transform according to certain rules under coordinate transformation.

• The term tensor was derived from the Latin word 'tenus', which means tension or stress. Tensor was first used for mathematical description of mechanical stress.

#### **Isotropic tensor:**

If the values of the **components** of a given tensor are invariant under coordinate transformation by proper rotation of axes, then the tensor is said to be an isotropic tensor. E.g. scalars (or zero rank tensors), first rank tensor of any dimension ( or zero or null vector), Kronecker Delta (or second rank tensor), Levi-Civita tensor (or third rank tensor).

#### Mixed tensor:

A mixed tensor has both contravariant and covariant characteristics. E.g.  $A_{j}^{i}, B_{jk}^{i}, C_{k}^{ij}$ , etc.

The minimum rank of a mixed tensor is **two**- one fold contravariant and one fold covariant, i.e.  $A_j^i$  and its transformation rule is given below.

$$x^{i} \xrightarrow{Coordinate Transformation} \overline{x}^{p}$$

$$A_{j}^{i} \xrightarrow{Coordinate Transformation} \overline{A}_{q}^{p}$$

$$\overline{A}_{q}^{p} = \sum_{i,j=1}^{N} \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial x^{j}}{\partial \overline{x}^{q}}\right) A_{j}^{i} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial x^{j}}{\partial \overline{x}^{q}}\right) A_{j}^{i}$$

The transformation rule for a mixed tensor of rank (m+n) with m-fold contravariant and n-fold covariant in N dimensionsis given below.

 $x^{i} \xrightarrow{Coordinate Transformation} \xrightarrow{\pi^{p}} x^{p}$ 

#### **Invariant:**

If a physical quantity remains unchanged under a coordinate transformation, then that quantity is said to be an invariant **under that particular transformation**. Scalars (or zero rank tensors) are invariants under any coordinate transformation.

#### **Problems in Transformation rules for tensors**

1. Which of the following quantities is a tensor of rank 1? (2016)

$$\sum_{i=1}^{3} a^{i} b_{i} , \quad \sum_{k=1}^{3} a^{ik} b_{k} , \quad \sum_{i,k=1}^{3} P_{ik} \xi^{i} \xi^{k}$$

Hints: Repeated index is for summation, no. of free inices gives us the rank.

2. A vector  $\vec{X}$  has components  $(x^1, x^2)$ . If the coordinate system is rotated counterclockwise by an angle  $\theta$ , the components are transformed into  $(\vec{x}, \vec{x})$ . From the transformation rule

$$\overline{x}^{i} = \sum_{j=1}^{2} \frac{\partial \overline{x}^{i}}{\partial x^{j}} x^{j}$$
tain the matrix form of  $\left(\frac{\partial \overline{x}^{i}}{\partial x^{j}}\right)$ . (2016)

Hints: i=1

Ob

$$i = 1, \qquad \bar{x}^1 = \sum_{j=1}^2 \frac{\partial \bar{x}^1}{\partial x^j} x^j = \frac{\partial \bar{x}^1}{\partial x^1} x^1 + \frac{\partial \bar{x}^1}{\partial x^2} x^2$$
$$i = 2, \qquad \bar{x}^2 = \sum_{j=1}^2 \frac{\partial \bar{x}^2}{\partial x^j} x^j = \frac{\partial \bar{x}^2}{\partial x^1} x^1 + \frac{\partial \bar{x}^2}{\partial x^2} x^2$$

$$\begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = a_{ij} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

Hence, matrix form of  $\frac{\partial \bar{x}^i}{\partial x^j}$  is

$$a_{ij} = \begin{pmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} \end{pmatrix}$$

- 3. Show that displacement, velocity and acceleration are 1<sup>st</sup> rank contravariant tensors. (See theory)
- Determine whether the acceleration is a contravariant or covariant tensor, mention its rank. (2017) (See theory)
- Give two physical examples of a 2<sup>nd</sup> rank tensor. (2015) (See theory)
- 6. Is  $B^{ij} = x^i x^j$  a tensor?

Hence,

Hints:

$$x^{i} \xrightarrow{Coordinate Transformation} \overline{x}^{p}$$

$$B^{ij} = x^{i}x^{j} \xrightarrow{Coordinate Transformation} \overline{B}^{pq} = \bar{x}^{p}\bar{x}^{q}$$

$$\overline{B}^{pq} = \bar{x}^{p}\bar{x}^{q} = \left(\frac{\partial \bar{x}^{p}}{\partial x^{i}}x^{i}\right)\left(\frac{\partial \bar{x}^{q}}{\partial x^{j}}x^{j}\right)$$

$$= \left(\frac{\partial \bar{x}^{p}}{\partial x^{i}}\right)\left(\frac{\partial \bar{x}^{q}}{\partial x^{j}}\right)x^{i}x^{j}$$

$$\overline{B}^{pq} = \left(\frac{\partial \bar{x}^{p}}{\partial x^{i}}\right)\left(\frac{\partial \bar{x}^{q}}{\partial x^{j}}\right)B^{ij}$$

Since the above relation is exactly similar to the transformation rule for  $2^{nd}$  rank contravariant tensor,  $B^{ij} = x^i x^j$  must be a contravariant tensor of rank 2.

Is  $x^{i}(1+x^{j})$  a tensor? Hints:  $B^{ij} = x^{i}(1+x^{j})$   $\overline{B}^{pq} = \overline{x}^{p}(1+\overline{x}^{q}) = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}x^{i}\right)\left(1+\frac{\partial \overline{x}^{q}}{\partial x^{j}}x^{j}\right)$  $= \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right)x^{i} + \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right)\left(\frac{\partial \overline{x}^{q}}{\partial x^{j}}\right)x^{i}x^{j}$ 

The above equation shows that  $B^{ij} = x^i (1+x^j)$  does not obey any transformation rule for tensors. So, it never be a tensor.

8. Show that  $\frac{\partial A_i}{\partial x^j}$  is not a tensor although  $A_i$  is an arbitrary covariant tensor of rank 1. (2017)

Hints: Since  $A_i$  is a contravariant tensor, its transformation rule is

$$\bar{A}_{p} = \frac{\partial x^{i}}{\partial \bar{x}^{p}} A_{i}$$
$$x^{i} \xrightarrow{Coordinate Transformation} \bar{x}^{p}$$

 $\frac{\partial A_i}{\partial x^j} \xrightarrow{Coordinate Transformation} \frac{\partial \bar{A}_p}{\partial \bar{x}^q}$ 

Hence,

$$\frac{\partial \bar{A}_p}{\partial \bar{x}^q} = \frac{\partial}{\partial \bar{x}^q} \left[ \frac{\partial x^i}{\partial \bar{x}^p} A_i \right] = \frac{\partial}{\partial \bar{x}^q} \left( \frac{\partial x^i}{\partial \bar{x}^p} \right) A_i + \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial A_i}{\partial \bar{x}^q}$$

Since, the above expression can not be moulded to the form of a transformation rule for any tensor,  $\frac{\partial A_i}{\partial x^j}$  never be a tensor.

9. In the language of tensors, what is the type of gradient of a scalar field? (2018) Hints: The transformation rule for the gradient of a scalar field (section: 4a)

$$\frac{\partial \phi}{\partial x^{p}} = \left(\frac{\partial x^{i}}{\partial x^{p}}\right) \frac{\partial \phi}{\partial x^{i}} \qquad i, p = 1, 2, 3, \dots, N$$

Which is similar to the transformation rule for 1<sup>st</sup> rank covariant tensor. So, gradient of a scalar field is a 1<sup>st</sup> rank covariant tensor.

10. If the components of a tensor are zero in coordinate system, then prove that the components are zero in all coordinate system.

Hint: 
$$\bar{A}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} A^{ij}$$
, if  $A^{ij} = 0$ , then  $\bar{S}^{pq} = 0$ .

11. The components of a contravariant vector in the x-coordinate system are 6 and 3. Obtain its components in the  $\overline{x}$ - coordinate system, if

$$\overline{x}^{1} = 7(x^{1})^{2}$$
 and  $\overline{x}^{2} = 6(x^{1})^{2} + 2(x^{2})^{2}$ 

Hints:

$$(x^1, x^2) \rightarrow (\bar{x}^1, \bar{x}^2)$$
  
 $A^1 = 6, A^2 = 6 \rightarrow \bar{A}^1 = ?, \quad \bar{A}^2 = ?$  (In new coordinate system( $\bar{x}^1, \bar{x}^2$ ))

Transformation rule

$$\bar{A}^p = \sum_{i=1}^2 \frac{\partial \bar{x}^p}{\partial x^i} A^i$$

$$p = 1, \qquad \bar{A}^{1} = \sum_{i=1}^{2} \frac{\partial \bar{x}^{1}}{\partial x^{i}} A^{i} = \frac{\partial \bar{x}^{1}}{\partial x^{1}} A^{1} + \frac{\partial \bar{x}^{1}}{\partial x^{2}} A^{2}$$
$$= \frac{\partial}{\partial x^{1}} \{7(x^{1})^{2}\} + \frac{\partial}{\partial x^{2}} \{6(x^{1})^{2} + 2(x^{2})^{2}\}$$

Hence,  $\bar{A}^1 = 84x^1 = 12\sqrt{7}\bar{x}^{1-1/2}$ , etc.

(Ans: 
$$\overline{A}^{1} = 12\sqrt{7} \left( \frac{x}{x} \right)^{-\frac{1}{2}}, \ \overline{A}^{2} = \frac{72}{\sqrt{7}} \left( \frac{x}{x} \right)^{-\frac{1}{2}} + 6\sqrt{2} \left[ \left( \frac{x}{x} \right)^{2} - \frac{6}{7} \frac{x}{x} \right]^{\frac{1}{2}}$$
)

12. Find the covariant components of a tensor in cylindrical coordinates  $\rho$ ,  $\phi$  and z, if its covariant components in rectangular coordinates x, y & z are 2x-z,  $x^2y$  and yz.

Hints:

$$i, p = 1, 2, 3$$

Coordinate Transformation

In old coordinate system  $(x^1 = x, x^2 = y, x^3 = z)$  $A_1 = 2x - z (= 2x^1 - x^3), \qquad A_2 = x^2 y (= (x^1)^2 x^2), \qquad A_2 = y z (= x^2 x^3)$ 

In new coordinate system ( $\bar{x}^1 = \rho, \bar{x}^2 = \varphi, \bar{x}^3 = z$ )  $\bar{A}_1 = ?$   $\bar{A}_2 = ?$   $\bar{A}_3 = ?$ 

Transformation rule

$$\bar{A}_p = \frac{\partial x^i}{\partial \bar{x}^p} A_i$$

Relations between the two sets of coordinates

$$\begin{aligned} \mathbf{x} &= \rho \cos\varphi, \quad \mathbf{y} = \rho \sin\varphi, \quad \mathbf{z} = \mathbf{z} \\ \text{If } p = \mathbf{1}, \ \bar{A}_1 &= \frac{\partial x^1}{\partial \bar{x}^1} A_1 = \frac{\partial x^1}{\partial \bar{x}^1} A_1 + \frac{\partial x^2}{\partial \bar{x}^1} A_2 + \frac{\partial x^3}{\partial \bar{x}^1} A_3 \\ &= \frac{\partial x}{\partial \rho} (2x - z) + \frac{\partial y}{\partial \rho} (x^2 y) + \frac{\partial z}{\partial \rho} (yz) \\ &= \left[ \frac{\partial}{\partial \rho} (\rho \cos\varphi) \right] (2\rho \cos\varphi - z) + \left[ \frac{\partial}{\partial \rho} (\rho \sin\varphi) \right] ((\rho \cos\varphi)^2 \rho \sin\varphi) \\ &+ \left[ \frac{\partial z}{\partial \rho} \right] (\rho \sin\varphi \times z) \\ \text{Hence,} \qquad \bar{A}_1 = 2\rho \cos^2\varphi - z \cos\varphi + \rho^3 \sin^2\varphi \cos^2\varphi \end{aligned}$$

Hence, 
$$A_1 = 2\rho \cos^2 \varphi - z \cos \varphi + \rho^3 \sin^2 \varphi \cos^2 \varphi$$
  
Similarly,  $\bar{A}_2 = -2\rho^2 \sin \varphi \cos \varphi - \rho z \sin \varphi + \rho^4 \sin \varphi \cos^3 \varphi$   
And  $\bar{A}_3 = \rho z \sin \varphi$ 

- 13. A covariant tensor has components xy,  $2y x^2$  and xz in rectangular coordinates, find its covariant components in spherical coordinates. (2017) Hints: Same as the Problem-11 Relations between the two sets of coordinates  $x = r \sin\theta \cos\varphi$ ,  $y = r\sin\theta\sin\varphi$ ,  $z = r \cos\theta$ (Ans:  $2r\sin^2\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\cos\phi$ ,  $2r^2\sin\theta\sin^2\phi\cos\theta - r^3\sin^2\theta\cos\theta\cos\phi$ and  $-r^3\sin^3\theta\cos\phi + 2r^2\sin^2\theta\sin\phi\cos\phi$ )
- 14. The Cartesian components of velocity vector of a fluid in motion in a two dimensional plane are given by  $v_x = x^2$  and  $v_y = y^2$ . Find the components of the velocity vector in  $(r, \theta)$  polar coordinates. (2018).  $(r,\theta)$ Hint: (x, y)Since,  $x = r \cos\theta$ ,  $y = r \sin\theta \rightarrow r = (x^2 + y^2)^{1/2}$ ,  $\theta =$ In tensorial notation  $x = x^1$ ,  $y = x^2$ and  $\bar{x}^1 = [(x^1)^2 + (x^2)^2]^{1/2}, \ \bar{x}^2 = tan^{-1}\frac{y}{x}$  $x^1 = \bar{x}^1 \cos \bar{x}^2, \ x^2 = \bar{x}^1 \cos \bar{x}^2$ Given, components of velocity vector in (x, y) system  $v_x = x^2$ ,  $v_y = y^2$ In tensorial notation, in  $(x^1, x^2)$  system  $v^1 = (x^1)^2$ ,  $v^2 = (x^2)^2$ In  $(\bar{x}^1, \bar{x}^2)$  system  $\overline{v^1}$ Transformation rule,  $\bar{v}^p = \sum_{i=1}^{2} \frac{\partial \bar{x}^p}{\partial x^i} v^i$  $\bar{v}^{1} = \sum_{i=1}^{2} \frac{\partial \bar{x}^{1}}{\partial x^{i}} v^{i} = \frac{\partial \bar{x}^{1}}{\partial x^{1}} v^{1} + \frac{\partial \bar{x}^{1}}{\partial x^{2}} v^{2}$ *p* = 1,  $\frac{\partial r}{\partial x}v_x + \frac{\partial r}{\partial y}v_y = \left\{\frac{\partial}{\partial x}(x^2 + y^2)^{1/2}\right\}x^2 + \left\{\frac{\partial}{\partial y}(x^2 + y^2)^{1/2}\right\}y^2$  $v_r = r^2(\cos^3\theta + \sin^3\theta)$ After simplification  $v_{\theta} = r(\cos\theta + \sin\theta)\sin\theta\cos\theta$ Similarly,
- 15. What is a mixed tensor? Write down all possible transformation rules for a mixed tensor of rank 4.

Hints:  $\bar{A}_{pq\ldots}^{\alpha\beta\ldots} = \left(\frac{\partial \bar{x}^{\alpha}}{\partial x^{a}}\frac{\partial \bar{x}^{\beta}}{\partial x^{b}}\ldots\right) \left(\frac{\partial x^{i}}{\partial \bar{x}^{p}}\frac{\partial x^{j}}{\partial \bar{x}^{q}}\ldots\right) A_{ij\ldots}^{ab\ldots}$ Examples of \$th rank mixed tensor:  $A_{i}^{jkl}, B_{ab}^{cd}, C_{pqr}^{s}$ 

16. Write down the transformation rules for a 4<sup>th</sup> rank mixed tensor of 1-fold covariant and 3-fold contravariant.

# 7. Kronecker Delta

It is a unit tensor (or unity tensor) of rank 2, whose all elements are zero except those with identical values of the two indices and those with identical values of the two indices are assigned the value equal to 1.

$$\delta_{j}^{i} = \frac{\partial x^{i}}{\partial x^{j}} = 1, \qquad i = j$$
$$= 0 \qquad i \neq j$$

i = i

Proof: Kronecker Delta is a tensor of rank 2

By definition

$$= 0 \qquad i \neq j$$

$$= 0 \qquad i \neq j$$

$$x^{i} \xrightarrow{Coordinate Transformation} \xrightarrow{\overline{x}^{p}} \overline{\delta_{q}^{p}}$$

$$\overline{\delta_{q}^{p}} = \frac{\partial \overline{x}^{p}}{\partial \overline{x}^{q}} = \frac{\partial}{\partial \overline{x}^{q}} \overline{x}^{p} \left(x^{1}, x^{2}, ..., x^{N}\right) = \sum_{i=1}^{N} \frac{\partial \overline{x}^{p}}{\partial x^{i}} \frac{\partial x^{i}}{\partial \overline{x}^{q}} \quad \text{(Using chain rule in differential calculus)}$$

$$= \frac{\partial \overline{x}^{p}}{\partial x^{i}} \frac{\partial x^{i}}{\partial \overline{x}^{q}} \qquad \text{(According to Eienstein's summation convention)}$$

$$= \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial x^{j}}{\partial \overline{x}^{q}}\right) \frac{\partial x^{i}}{\partial x^{j}}$$

$$\therefore \quad \overline{\delta}_{q}^{p} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial x^{j}}{\partial \overline{x}^{q}}\right) \delta_{j}^{i}$$

This equation is exactly similar to the transformation rule for a mixed 2<sup>nd</sup> rank tensor of 1-fold contravariant 1-fold covariant.

(We may start with  $\delta^{ij}$  and  $\delta_{ij}$ , and then also we will find a transformation rule for 2<sup>nd</sup> rank tensor.)

Proof: Kronecker Delta is an invariant in all coordinate systems, and hence isotropic.

By definition

If

It is

Cov

$$\delta_{j}^{i} = \frac{\partial x^{i}}{\partial x^{j}} = 1, \qquad i = j$$
$$= 0 \qquad i \neq j$$

We start with the transformation rule for Kronecker Delta.

$$\overline{\delta}_{q}^{p} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial x^{j}}{\partial \overline{x}^{q}}\right) \delta_{j}^{i}$$

$$= \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial x^{j}}{\partial \overline{x}^{q}}\right) \frac{\partial x^{i}}{\partial x^{j}} \text{ (Applying the definition of Kronecker delta)}$$

$$= \frac{\partial \overline{x}^{p}}{\partial x^{i}} \frac{\partial x^{i}}{\partial \overline{x}^{q}}$$

$$\therefore \quad \overline{\delta}_{q}^{p} = \frac{\partial \overline{x}^{p}}{\partial \overline{x}^{q}}$$

$$p = q, \qquad \overline{\delta}_{q}^{p} = 1$$

$$p \neq q \qquad \overline{\delta}_{q}^{p} = 0.$$
a symmetric tensor, as  $\delta^{ij} = \delta^{ji}$ ,  $\delta_{ij} = \delta_{jj}$ ,  $\delta_{i}^{j} = \delta_{j}^{i}$ 
ariant, contravariant and mixed type of this tensor are the same, i.e.  $\delta^{ij} = \delta_{ij} = \delta_{ij}^{j}$ 

#### **Problems in Kronecker Delta**

- 1. Prove that Kronecker Delta  $\delta_j^i$  is a mixed tensor of rank two. (See theory)
- 2. Show that the components  $\delta_j^i$  of have the same values in all coordinate systems. (2014) (see theory)

- 3. What is the value of Kronecker Delta  $\delta_{21}^{12}$ ? (2017) (Hint: 0, apply the property of Kronecker Delta  $\delta_j^i$ , when  $i \neq j$ .(i=12, j=21))
- 4. What is the value of  $\delta_i^i$  in six dimensional space. (2018) Hints:  $\delta_i^i = \sum_{i=1}^6 \delta_i^i = \delta_1^1 + \delta_2^2 + \dots + \delta_6^6 = 1 + 1 + \dots + 1 = 6.$
- 5. Evaluate the quantity  $\sum_{i,j=1} \delta_j^i$  in four dimensional coordinate system. (2014)

Hints: 
$$\sum_{i,j=1}^{4} \delta_j^i = \sum_{j=1}^{4} \left( \delta_j^1 + \delta_j^2 + \delta_j^3 + \delta_j^4 \right)$$
$$= \left( \delta_1^1 + \delta_1^2 + \delta_1^3 + \delta_1^4 \right) + \left( \delta_2^1 + \delta_2^2 + \delta_2^3 + \delta_2^4 \right) + \left( \delta_3^1 + \delta_3^2 + \delta_3^3 + \delta_3^4 \right)$$
$$+ \left( \delta_4^1 + \delta_4^2 + \delta_4^3 + \delta_4^4 \right)$$
$$= \left( 1 + 0 + 0 + 0 \right) + \left( 0 + 1 + 0 + 0 \right) + \left( 0 + 0 + 1 + 0 \right) + \left( 0 + 0 + 1 \right) = 4.$$

6. Evaluate

$$\sum_{j=1,2,3} \delta_j^1 A^j . \qquad (201)$$

(Ans:  $A^1$ )

- 7. Evaluate  $\delta_j^i \delta_k^j \delta_l^k \delta^l$  in N-dimensional space. (2018) Hints:  $\delta_j^i \delta_k^j \delta_l^k \delta^l = \delta_j^i \delta_k^j (\delta_l^k \delta^l) = \delta_j^i \delta_k^j \delta^k$ , etc.
- 8. Show that  $\delta_{j}^{i}U^{i}U^{j}$  represents square of the magnitude of the vector  $\vec{U} = \hat{i}U^{1} + \hat{j}U^{2} + \hat{k}U^{3}$ . (2016) Hints:  $\delta_{j}^{i}U^{i}U^{j} = \sum_{i,j=1}^{3}\delta_{j}^{i}U^{i}U^{j} = \sum_{j=1}^{3}\delta_{j}^{j}U^{1}U^{j} + \delta_{j}^{2}U^{2}U^{j} + \delta_{j}^{3}U^{3}U^{j}$   $= (\delta_{1}^{1}U^{1}U^{1} + \delta_{1}^{2}U^{2}U^{1} + \delta_{1}^{3}U^{3}U^{1}) + (\delta_{2}^{1}U^{1}U^{2} + \delta_{2}^{2}U^{2}U^{2} + \delta_{2}^{3}U^{3}U^{2})$   $+ (\delta_{3}^{1}U^{1}U^{3} + \delta_{3}^{2}U^{2}U^{3} + \delta_{3}^{3}U^{3}U^{3})$  $= U^{1}U^{1} + U^{2}U^{2} + U^{3}U^{3} = \vec{U}.\vec{U} = |\vec{U}|^{2}$

9. Show that  $\delta_i^j U^i V^j$  behaves like dot product of  $\vec{U} = \hat{i}U^1 + \hat{j}U^2 + \hat{k}U^3$  and  $\vec{V} = \hat{i}V^1 + \hat{j}V^2 + \hat{k}V^3$ . (2015) (same as the problem-8)

10. Show that

$$\left(\delta_m^i \delta_p^k + \delta_p^i \delta_m^k\right) W_{ik} = W_{mp} + W_{pm}.$$
(2015)  
Hints:  $\delta_m^i \left(\delta_p^k W_{ik}\right) + \delta_p^i \left(\delta_m^k W_{ik}\right) = \delta_m^i W_{ip} + \delta_j^i W_{im} = .....$ 

11. If  $x = x^1$ ,  $y = x^2$ ,  $z = x^3$ , show that

$$\sum_{i,j=1,2,3} \frac{\partial^2 \phi}{\partial x^i \partial x^j} \delta^i_j = \nabla^2 \phi.$$
 (2015)

# 8. Symmetric and anti-symmetric tensor tensors

#### Symmetric tensors

If any two indices of a tensor are interchanged and nothing changes, then the tensor is called symmetric tensor w.r.t. those two indices.

E.g. If  $A_k^{ij}$  be a 3<sup>rd</sup> rank tensor and  $A_k^{ij} = A_k^{ji}$  for all i and j,  $A_k^{ij}$  is said to be a symmetric tensor w.r.t. i and j.

We consider a symmetric tensor  $S_{ij}$  w.r.t. i and j in 3D coordinate system. Its 9 components are given below.

$$S_{ij} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$$

Since  $S_{ji} = S_{ij} \rightarrow S_{21} = S_{12}$ ,  $S_{32} = S_{23}$ ,  $S_{31} = S_{13}$ , the total number of independent components of a 2<sup>nd</sup> rank tensor in 3D is 6.

$$S_{ij} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix}$$

• The number of independent components of a Rth rank symmetric tensor in N dimensional space is equal to the number of combination of (N+R-1) taking R

at a time, i.e. 
$${}^{N+R-1}C_R = \frac{(N+R-1)!}{R!(N-1)!}$$

#### Anti-symmetric (or skew-symmetric) tensors

If any two indices of a tensor are interchanged and a –ve sign comes out, then the tensor is said to be an anti-symmetric tensor w.r.t. those two indices.

E.g. If  $B_{lm}^{ijk}$  be a 5<sup>th</sup> rank tensor and  $B_{lm}^{ikj} = -B_{lm}^{ijk}$  for all j and k, then  $B_{lm}^{ijk}$  is said to be an anti-symmetric tensor w.r.t. k and l.

We consider an anti-symmetric tensor  $A_{ij}$  w.r.t. i and j in 3D coordinate system. Its 9 components are given below.

$$A_{ij} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

Since  $A_{ii} = -A_{ii} \rightarrow A_{21} = -A_{12}$ ,  $A_{32} = -A_{23}$ ,  $A_{31} = -A_{13}$ , and  $A_{11} = A_{22} = A_{333} = 0$ , (Since  $A_{ij} = -A_{ji} \rightarrow A_{ij} + A_{ji} = 0 \rightarrow if \ i = j, then \ 2A_{ii} = 2A_{jj} = 0.$ ) SO

$$A_{ij} = \begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix}$$

The number of independent components of a Rth rank ant-symmetric tensor in N dimensional space is equal to the number of combination of (N+R-1) taking

R at a time, i.e. 
$${}^{N}C_{R} = \frac{N!}{R!(N-R)!}$$

For skew-symmetric, if N < R, what will happen? For example, N=2 and R=3 for a tensor  $C_{ijk}$ , of which the components are

$$C_{111}, C_{112}, C_{121}, C_{122}, C_{211}, C_{221}$$
 and  $C_{222}$ . But  
 $C_{111} = C_{112} = C_{121} = C_{122} = C_{211} = C_{221} = C_{222} = 0$ , so N must be larger than R for  
anti-symmetric tensor.

- The indices whose exchange defines the symmetry and anti-symmetry relations should be of the same variance type, i.e. both upper or both lower.
- The symmetry and anti-symmetry characteristic of a tensor is invariant under • coordinate transformation.

# Proof: Every tensor can be decomposed into a symmetric and an anti-symmetric part.

We consider a covariant tensor  $A_{ij}$  of rank two.

$$A_{ij} = \frac{1}{2}A_{ij} + \frac{1}{2}A_{ij} = \left(\frac{1}{2}A_{ij} + \frac{1}{2}A_{ji}\right) + \left(\frac{1}{2}A_{ij} - \frac{1}{2}A_{ji}\right) = P_{ij} + Q_{ij}$$
  
where  $P_{ij} = \frac{1}{2}A_{ij} + \frac{1}{2}A_{ji}$  and  $Q_{ij} = \frac{1}{2}A_{ij} - \frac{1}{2}A_{ji}$ 

If  $i \square j$ , then  $P_{ij} \rightarrow P_{ji} = \frac{1}{2}A_{ji} + \frac{1}{2}A_{ij} = P_{ij}$ , and so,  $P_{ij}$  is a symmetric tensor w.r.t. the indices i and j.

If  $i \square j$ , then  $Q_{ij} \to Q_{ji} = \frac{1}{2}A_{ji} - \frac{1}{2}A_{ij} = -Q_{ij}$ , and so  $Q_{ij}$  is an anti-symmetric tensor w.r.t. the indices i and j.

Thus every tensor can be decomposed into a symmetric tensor and an anti-symmetric tensor.

#### Problems in Symmetric and anti-symmetric tensors

1. Show that every tensor can be decomposed into a symmetric and an anti-symmetric part.

(See theory)

- What is the number of independent components of anti-symmetric tensor A<sup>mn</sup> in four dimensions? (2018) Hints: <sup>N=4</sup>C<sub>R=2</sub>=6
- 3. If  $A_{ij}$  is a symmetric tensor, which of the following is correct? (2014)

(i) 
$$A_{ij} + A_{ji} = 0$$

(ii)  $A_{ii} + A_{ii} = 2A_{ii}$ 

(iii) 
$$A_{ii} - A_{ii} \neq 0$$

(iv) None of these

(Ans: (ii))

4. Show that symmetry / anti-symmetry of a tensor is maintained under coordinate transformation.

Hints: Let  $S^{ij}$  be a symmetric tensor. So,  $S^{ji} = S^{lj}$ 

Transformation rule

$$\bar{S}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} S^{ij}$$

$$= \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial \bar{x}^p}{\partial x^i} S^{ij}$$

$$= \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial \bar{x}^p}{\partial x^i} S^{ji}$$

$$= \bar{S}^{qp}$$

Thus,  $\bar{S}^{pq}$  is again a symmetric tensor in the new coordinate system. Similarly we can show that if the tensor is anti-symmetric, i.e.  $S^{ji} = -S^{ij}$ , the property will remain unchanged in the new coordinate system, i.e.  $\bar{S}^{pq} = -\bar{S}^{qp}$ .

 Under transformation coordinate, mention whether anti-symmetric property of a mixed tensor is conserved or not. Explain with reason. (2018) Hints: Not permissible.

$$\bar{A}_{q}^{p} = \frac{\partial \bar{x}^{p}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{q}} A_{j}^{i}$$

$$p \leftrightarrow q, \quad \bar{A}_{p}^{q} = \frac{\partial \bar{x}^{q}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{p}} A_{j}^{i} = \frac{\partial \bar{x}^{q}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{p}} (-A_{i}^{j}) = -\frac{\partial x^{j}}{\partial \bar{x}^{p}} \frac{\partial \bar{x}^{q}}{\partial x^{i}} A_{i}^{j}$$

The RHS of the above equation can not reproduce  $-\bar{A}_q^p$ , as the cofactors  $\frac{\partial x^j}{\partial \bar{x}^p}$  and  $\frac{\partial \bar{x}^q}{\partial x^i}$  are not in the correct form. So, anti-symmetric property is not observed in the transformed components in the new coordinate system.

(Same case will happen for symmetric property of a mixed tensor.)

- 6.  $g_{ij}$  is a symmetric covariant tensor in 3-dimension (*i*, *j* = 1,2,3). Expressing it in a matrix form, show that it has 6 independent components. (2015) (See theory)
- 7. Show that  $A_{ij} = \frac{\partial a_i}{\partial x^j} \frac{\partial a_j}{\partial x^i}$  is an anti-symmetric tensor. (2016) (Hint: See  $i \square j$ )
- 8. If  $A_{ij}$  is a skew-symmetric tensor, prove that

$$\left(\delta_{j}^{i}\delta_{l}^{k}+\delta_{l}^{i}\delta_{j}^{k}\right)A_{ik}=0. (2016)$$

- 9. If  $A^{mn}$  is an anti-symmetric tensor and  $B_m$  is a vector, show that  $A^{mn}B_mB_n = 0$ . (2017) Hints:  $A^{mn}B_mB_n = A^{nm}B_nB_m$   $(m \leftrightarrow n) = -A^{mn}B_nB_m = -A^{mn}B_mB_n$ Hence,  $2A^{mn}B_mB_n = 0 \rightarrow A^{mn}B_mB_n = 0$
- 10. If  $A_{\lambda\mu}$  is a skew-symmetric tensor, show that

$$\left(B_{\nu}^{\mu}B_{\tau}^{\sigma}+B_{\tau}^{\mu}B_{\nu}^{\sigma}\right)A_{\mu\sigma}=0$$
 (2017)

11. If  $a_{ik}$  and  $b_{ik}$  are two symmetric tensors satisfying the equation

$$a_{ij}b_{kl} - a_{il}b_{jk} + a_{jk}b_{il} - a_{kl}b_{ij} = 0,$$

show that  $a_{ij} = \rho b_{ij}$  is a solution of the above equation where  $\rho$  is a constant. (2016) Hints: LHS= $\rho b_{ij}b_{kl} - \rho b_{il}b_{jk} + \rho b_{jk}b_{il} - \rho b_{kl}b_{ij} = 0.$ 

12. If  $A_{ij}$  is a symmetric and  $B^{ij}$  is an anti-symmetric tensors, prove that  $A_{ij}B^{ij} = 0$ .

13. If  $\vec{A}$  and  $\vec{B}$  are two ordinary vectors, then show that components of  $\vec{A} \times \vec{B}$  form a second rank anti-symmetric tensor. (2014)

Hints: Let,  $\vec{A} = (A_1, A_2, A_3)$  and  $\vec{B} = (B_1, B_2, B_3)$  in 3D.  $(\vec{A}X\vec{B})_1 = A_2B_3 - A_3B_2 = T_{23}$ 

Similarly

$$(\vec{A}X\vec{B})_2 = A_3B_2 - A_2B_3 = T_{32}$$
  
 $(\vec{A}X\vec{B})_3 = A_1B_2 - A_2B_1 = T_{12}$ 

Hence,

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$$I_{ij} = A_i B_j - A_j B_i$$
  
$$i \leftrightarrow j, \quad T_{ji} = A_j B_i - A_i B_j = -(A_i B_j - A_j B_i) = -T_{ij} \text{ (anti-symmetry)}$$

And

$$T_{ii} = T_{jj} = 0$$

$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} 0 & T_{12} & T_{13} \\ -T_{12} & 0 & T_{23} \\ -T_{13} & -T_{23} & T_{33} \end{pmatrix}$$

Thus, number of independent components  $T_{ij}$  of  $\vec{A} \times \vec{B}$  is 3 in 3D. Hence,  $\vec{A} \times \vec{B}$  is a 2<sup>nd</sup> rank anti-symmetric tensor.

Alternately,

$$\begin{split} \bar{T}_{pq} &= \bar{A}_p \bar{B}_q - \bar{A}_q \bar{B}_p = \left(\frac{\partial x^i}{\partial \bar{x}^p} A_i\right) \left(\frac{\partial x^j}{\partial \bar{x}^q} B_j\right) - \left(\frac{\partial x^j}{\partial \bar{x}^q} A_j\right) \left(\frac{\partial x^i}{\partial \bar{x}^p} B_i\right) = \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} (A_i B_j - A_j B_i) \\ \bar{T}_{pq} &= \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} T_{ij}, \end{split}$$

which shows that the components of  $\vec{A} \times \vec{B}$  respect the transformation rule of 2<sup>nd</sup> rank tensor.

# PART - II

### **Rules of combination for tensors**

The algebraic operations defined on tensors are **addition**, **subtraction**, **multiplication** and **contraction**. Multiplication of tensors of two types - (i) Outer multiplication and (ii) Inner multiplication. The operation of division is undefined for tensors, but we have an algorithm-called the Quotient Law, which is similar to the division algorithm.

#### 1) Addition and Subtraction

**Rule:** The operations of addition and subtraction are defined only for tensors of same rank and type (i.e. contravariant and covariant in nature).

In the process of addition (and subtraction), the corresponding components of two tensors of the same number 'n' of contravariant indices and the same number 'm' of covariant indices are added together (and subtracted) and the quantities thus obtained are again the components of tensor of 'n' contravariant and 'm' covariant indices. i.e.

$$C_{jk}^{i} + D_{jk}^{i} = A_{jk}^{i}$$
  $F_{n}^{lm} + G_{n}^{lm} = S_{n}^{lm}$ 

(It is similar to addition and subtraction for vectors of same type and in component-wise manner.)

**Proof:**  $C_{jk}^{i}$  and  $D_{jk}^{i}$  are the components of two mixed tensor of same rank and type (1-fold contravariabt and 2-fold covariant 3<sup>rd</sup> rank tensor) in the  $x^{i}$  (i = 1, 2, 3, ..., N)-coordinate system of N dimensions. Now we consider a coordinate transformation from  $x^{i}$  (i = 1, 2, 3, ..., N)-coordinate system of N dimensions to  $\overline{x}^{p}$  (p = 1, 2, 3, ..., N)-coordinate system of N dimensions as follows.

$$\begin{array}{ccc} x^{i} & & \hline Coordinate \ Transformation & & \overline{x}^{p} \\ & & & & \\ C^{i}_{jk} & & & & \\ & & & & \\ D^{i}_{jk} & & & & \\ & & & & \\ D^{i}_{jk} & & & & & \\ \end{array} \xrightarrow{Coordinate \ Transformation & & & \\ & & & & & \\ D^{p}_{qi} & & & & \\ \end{array}$$

By the transformation rules

$$\overline{C}_{qr}^{p} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial x^{j}}{\partial \overline{x}^{q}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) C_{jk}^{i}$$
$$\overline{D}_{qr}^{p} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial x^{j}}{\partial \overline{x}^{q}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) D_{jk}^{i}$$

Hence,

$$\overline{C}_{qr}^{p} + \overline{D}_{qr}^{p} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial x^{j}}{\partial \overline{x}^{q}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) \left[C_{jk}^{i} + D_{jk}^{i}\right]$$

The above equation is exactly similar to the transformation rule for a mixed tensor of rank 3 with 1-fold contravariabt and 2-fold covariant. So,  $C_{jk}^i + D_{jk}^i$  must be a mixed tensor of rank 3 with 1-fold contravariabt and 2-fold covariant and the above addition rule is admissible for tensors.

(Similarly we can also prove the same rule for subtraction also......HW)

#### 2) Outer Multiplication of tensors

**Rule:** Two tensors can be multiplied component-wise to get a new tensor whose rank is the sum of that of the two tensors in the product. i.e.

 $A^i \rightarrow$  a contravariant tensor of rank 1 and

 $B_k^j \rightarrow$  a mixed tensor of rank 2 with 1-fold contravariant and 1-fold covariant,

Outer multiplication  $A^i \times B_k^j = C_k^{ij}$ , where  $C_k^{ij}$  is a mixed tensor of rank 3 (=1+2) with 2-fold contravariant and 1-fold covariant.

#### **Proof:**

$$x^{i} \xrightarrow{Coordinate Transformation} \overline{x}^{p}$$

$$A^{i} \xrightarrow{Coordinate Transformation} \overline{A}^{p}$$

$$B_{k}^{j} \xrightarrow{Coordinate Transformation} \overline{B}_{r}^{q}$$

$$i, j, k = 1, 2, 3, \dots, N \qquad p, q, r = 1, 2, 3, \dots, N$$

Applying transformation rules

$$\overline{A}^{p} \times \overline{B}^{q}_{r} = \left[ \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) A^{i} \right] \times \left[ \left( \frac{\partial \overline{x}^{q}}{\partial x^{j}} \right) \left( \frac{\partial x^{k}}{\partial \overline{x}^{r}} \right) B^{j}_{k} \right]$$
$$= \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) \left( \frac{\partial \overline{x}^{q}}{\partial x^{j}} \right) \left( \frac{\partial x^{k}}{\partial \overline{x}^{r}} \right) A^{i} \times B^{j}_{k}$$
$$\therefore \quad \overline{C}^{pq}_{r} = \left( \frac{\partial \overline{x}^{p}}{\partial x^{i}} \right) \left( \frac{\partial \overline{x}^{q}}{\partial x^{j}} \right) \left( \frac{\partial x^{k}}{\partial \overline{x}^{r}} \right) C^{ij}_{k}$$

The above equation is exactly similar to the transformation rule for a mixed tensor of rank 3 with 2-fold contravariabt and 1-fold covariant. So,  $C_k^{ij}$  must be a mixed tensor of rank 3 with

2-fold contravariabt and 1-fold covariant and the above multiplication rule is admissible for tensors.

(Prove the multiplication rule with tensors of different rank and type......HW)

#### 3) Contraction

**Rule:** In any **mixed** tensor, if we put a contravariant index equal to a covariant index, then summation over the repeated index is to be taken according to Einstein's summation convention and the **rank of the resultant tensor reduces by 2**. i.e.

 $C_{lm}^{ijk} \rightarrow$  a mixed tensor of rank 5 with 3-fold contravariant and 2-fold covariant

Applying contraction  $l = i \rightarrow C_{im}^{ijk} = \sum_{i=1}^{N} C_{im}^{ijk} = C_{1m}^{1jk} + C_{2m}^{2jk} + \dots + C_{Nm}^{Njk}$ 

(By Einstein's summation convention in N dimensional coordinate system)

 $C_{im}^{ijk} \left(= C_{1m}^{1jk} + C_{2m}^{2jk} + \dots + C_{Nm}^{Njk}\right)$  is a resultant tensor of rank 3 (=5-2) with 2-fold contravariant and 1-fold covariant. Index 'i' governs the summation, it has no role at all in determining the rank of the resultant tensor.

• Contraction can be repeatedly applied and in each contraction, the rank of the resultant tensor will reduce by 2. Applying  $2^{nd}$  contraction m = j in the above tensor

 $C_{im}^{ijk}$ , the resultant tensor becomes  $C_{ij}^{ijk} = \sum_{i,j=1}^{N} C_{ij}^{ijk}$ , rank of which is 1 (=5-2-2).

Here 'i' and 'l', and 'j' and 'm' are called **contractable indices**. The other pairs of contractable indices are (i,m), (j,l), (k,l) and (k,m).

- A given tensor with 'm' contravariant indices and 'n' covariant indices can be contracted in **m x n different ways**.
- But (i,j), (i,k), (j,k) and (l,m) are not the pairs of contractable indices. If the contraction is done, the resultant quantity will no longer be a tensor.

1.

**Proof:** If contraction is applied once, the resulting sum is a tensor of rank two less than that of the original tensor.

$$\begin{array}{c} x^{i} & \xrightarrow{Coordinate Transformation} & \overline{x}^{p} \\ C_{k}^{ij} & \xrightarrow{Coordinate Transformation} & \overline{C}_{r}^{pq} \\ i, j, k = 1, 2, 3, \dots, N \qquad p, q, r = 1, 2, 3, \dots, N \end{array}$$

By transformation rules of mixed tensor of rank 3 with 2-fold contravariant and 1-fold covariant

$$\overline{C}_{r}^{pq} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial \overline{x}^{q}}{\partial x^{j}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) C_{k}^{ij}$$

Applying contraction

$$r = q \quad \rightarrow \qquad \overline{C}_{q}^{pq} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial \overline{x}^{q}}{\partial x^{j}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{q}}\right) C_{k}^{ij}$$
$$= \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial x^{k}}{\partial x^{j}}\right) C_{k}^{ij}$$
$$= \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \delta_{j}^{k} C_{k}^{ij}$$

By definition, Kronecker Delta

$$\therefore \quad \overline{C}_{q}^{pq} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) C_{j}^{ij} = \sum_{j=1}^{N} \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) C_{j}^{ij}$$

The above equation is exactly similar to the transformation rule for a resultant contravariant tensor of rank 1 (j is the repeated index). So, the resulting sum must be a contravariant tensor of rank 1 (=3 - 2) and (q, r) is a pair of contractable indices. Thus the above contraction rule is admissible in mixed tensors.

 $\delta_j^i = \frac{\partial x^i}{\partial x^j} = 1,$ 

= 0

If possible, we apply the contraction q = p in  $\overline{C}_r^{pq}$ .

$$\overline{C}_{r}^{pp} = \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) \left(\frac{\partial \overline{x}^{p}}{\partial x^{j}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{q}}\right) C_{k}^{ij}$$

The above equation does not match with any transformation rules for tensors, therefore the above contraction is not admissible in case of tensors.

# 4) Inner multiplication

**Rule:** The inner product of two tensors means an outer product of the two followed by at least one contraction w.r.t. an index of one tensor and an index of **opposite** character of the **other** tensor. Thus, the inner product of two tensors is a tensor of rank two less than the sum of the ranks of the original two tensors. i.e.

 $A'_{ik} \rightarrow$  a mixed tensor of rank 3 with 1-fold contravariant and 2-fold covariant, and

 $B_l^m \rightarrow$  a mixed tensor of rank 2 with 1-fold contravariant and 1-fold covariant.

Step 1: The outer product of the two tensors is

$$A^i_{jk} \times B^m_l = C^{im}_{jkl},$$

where  $C_{jkl}^{im}$  is a mixed tensor of rank 5(=3+2) with 2-fold contravariant and 3-fold covariant.

**<u>Step 2</u>**: Now Applying contraction for two indices of opposite character and taking one from each of the two tensors.

$$l = i \rightarrow C_{jki}^{im} = \sum_{i=1}^{N} C_{jki}^{im} = C_{jk1}^{1m} + C_{jk2}^{2m} + \dots + C_{jkN}^{Nm}$$
, which is a resultant tensor of rank 3

(=5-2) with 2-fold contravariant and 1-fold covariant.

The second possible contraction is m=j or m=k.

#### 5) Quotient rule:

If a quantity X is inner-multiplied by an arbitrary tensor and the resulting quantity is again a tensor, then the quantity X is also a tensor.

It is a simplified rule to decide whether an unknown quantity is really a tensor or not.

#### **Statement:**

If a quantity X(i, j, k) is such that

$$X(i,j,k)A^i=B^j_k,$$

where  $A^i$  is an arbitrary contravariant tensor of rank 1 and  $B^j_k$  is a tensor of rank 2 with 1-fold contravariant and 1-fold covariant, then the quantity X(i, j, k) is a tensor of the type  $X^j_{ik}$ .

#### **Proof:**

$$x^{i} \xrightarrow{Coordinate Transformation} \overline{x}^{p}$$

$$X(i, j, k) \xrightarrow{Coordinate Transformation} \overline{X}(p, q, r)$$

$$A^{i} \xrightarrow{Coordinate Transformation} \overline{A}^{p}$$

$$B^{j}_{k} \xrightarrow{Coordinate Transformation} \overline{B}^{q}_{r}$$

$$i, j, k = 1, 2, 3, \dots, N \qquad p, q, r = 1, 2, 3, \dots, N$$

$$\therefore \overline{X}(p, q, r) \overline{A}^{p} = \overline{B}^{q}_{r}$$

$$\Rightarrow \overline{X}(p, q, r) \times \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) A^{i} = \left(\frac{\partial \overline{x}^{q}}{\partial x^{j}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) B^{j}_{k}$$

(By applying transformation rules for tensors)

$$= \left(\frac{\partial \overline{x}^{q}}{\partial x^{j}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) X(i, j, k) A^{i}$$

$$\Rightarrow \overline{X}(p, q, r) \left(\frac{\partial \overline{x}^{p}}{\partial x^{i}}\right) = \left(\frac{\partial \overline{x}^{q}}{\partial x^{j}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) X(i, j, k)$$

$$\Rightarrow \overline{X}(p, q, r) \left(\frac{\partial \overline{x}^{p}}{\partial \overline{x}^{i}}\right) \times \left(\frac{\partial x^{i}}{\partial \overline{x}^{s}}\right) = \left(\frac{\partial \overline{x}^{q}}{\partial x^{j}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) X(i, j, k) \times \left(\frac{\partial x^{i}}{\partial \overline{x}^{s}}\right)$$

$$\Rightarrow \overline{X}(p, q, r) \left(\frac{\partial \overline{x}^{p}}{\partial \overline{x}^{s}}\right) = \left(\frac{\partial \overline{x}^{q}}{\partial x^{i}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) \left(\frac{\partial x^{i}}{\partial \overline{x}^{s}}\right) X(i, j, k)$$

$$\Rightarrow \overline{X}(p, q, r) \overline{\delta}_{s}^{p} = \left(\frac{\partial \overline{x}^{q}}{\partial x^{i}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) \left(\frac{\partial x^{i}}{\partial \overline{x}^{s}}\right) X(i, j, k)$$

$$\Rightarrow \overline{X}(s, q, r) = \left(\frac{\partial \overline{x}^{q}}{\partial x^{i}}\right) \left(\frac{\partial x^{k}}{\partial \overline{x}^{r}}\right) \left(\frac{\partial x^{i}}{\partial \overline{x}^{s}}\right) X(i, j, k)$$
(Applying the constant of Kurner law Deltate)

(Applying the property of Kronecker Delta)

The above equation is exactly similar to the transformation rule for a mixed tensor of rank 3 with 1-fold contravariant and 2-fold covariant. So,  $X(i, j, k) = X_{ik}^{j}$  must be a mixed tensor of rank 3 with 1-fold contravariant and 2-fold covariant.

# **Problems in Combination Rules for Tensors**

# (Addition, Subtraction, Contraction, Outer multiplication, Inner multiplication)

- 1. Prove that the sum of two tensors of same rank and type is also a tensor. (See theory)
- 2. Prove that the difference of two tensors of same rank and type is also a tensor. (See theory)
- 3. Define outer product and inner product of two tensors. (2014)
- 4. Show that  $(A^i + B_i)$  is not a tensor.

Hints: 
$$x^{i} \xrightarrow{Coordinate Transformation} \overline{x}^{p}$$
  
 $(A^{i} + B_{i}) \xrightarrow{Coordinate Transformation} \overline{A}^{p} + \overline{B}_{p}$ 

Transformation rule

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i, \qquad \bar{B}_p = \frac{\partial x^i}{\partial \bar{x}^p} B_i$$

Hence,

$$\bar{A}^p + \bar{B}_p = \frac{\partial \bar{x}^p}{\partial x^i} A^i + \frac{\partial x^i}{\partial \bar{x}^p} B_i$$

The above expression shows that  $\overline{A}^p + \overline{B}_p$  does not respect transformation rule for any tensor, so  $(A^i + B_i)$  never be a tensor.

- 5. Show that  $(P^i + Q_{ij})$  is not a tensor.
- 6. If  $G^{ij}$  and  $T^{ij}$  are two contravariant tensors of rank 2 and satisfy the equation  $G^{ij} KT^{ij} = 0$  in coordinate system  $x^i$ , show that they satisfy the same form of equation  $\overline{G}^{ij} K\overline{T}^{ij} = 0$  under coordinate transformation  $x^i \to \overline{x}^i$ . Here K is an invariant. (2016)

Hints:  

$$\begin{aligned}
x^{i} \to x^{i} \\
G^{ij} - KT^{ij} &\to \overline{G}^{pq} - \overline{K}\overline{T}^{pq} = \overline{G}^{pq} - K\overline{T}^{pq} \\
\overline{G}^{pq} - K\overline{T}^{pq} &= \frac{\partial \overline{x}^{p}}{\partial x^{i}} \frac{\partial \overline{x}^{q}}{\partial x^{j}} G^{ij} - K \frac{\partial \overline{x}^{p}}{\partial x^{i}} \frac{\partial \overline{x}^{q}}{\partial x^{j}} T^{ij} = \frac{\partial \overline{x}^{p}}{\partial x^{i}} \frac{\partial \overline{x}^{q}}{\partial x^{j}} [G^{ij} - KT^{ij}] \\
\text{Hence, } \overline{G}^{pq} - K\overline{T}^{pq} &= \frac{\partial \overline{x}^{p}}{\partial x^{i}} \frac{\partial \overline{x}^{q}}{\partial x^{j}} \times 0 = 0
\end{aligned}$$

- 7. If  $A^i + B^i = 0$  in a coordinate system  $x^i$ , show that  $A'^j + B'^j = 0$  after a coordinate transformation  $x^i \to x'^i$ . (2015)
- 8. What do you mean by contraction of a mixed tensor? Contract  $A_{klm}^{ij}$  twice. (2014) (See theory)
- If contraction is applied once, prove that the resulting sum is a tensor of rank two less than that of the original tensor. (See theory)
- 10. If  $T_{mn}^{ij} = A^{ij}B_{mn}$  is a mixed tensor of rank four contracted by outer multiplication of  $A^{ij}$ and  $B_{mn}$ . Construct a tensor of rank 0 by the process of contraction. (2015) Hints: Applying contraction  $m = i \rightarrow T_{in}^{ij} = A^{ij}B_{in} = \sum_i A^{ij}B_{in}$ , there are 2 free indices, so rank of  $T_{in}^{ij}$  is 2.

Again applying contraction  $n = j \rightarrow T_{ij}^{ij} = A^{ij}B_{ij} = \sum_{ij} A^{ij}B_{ij}$ , there is not a single free indice, so rank of  $T_{ij}^{ij}$  is 0.

- 11. If a tensor of rank N is contracted 2 times, what would be its final rank? Obtain a zero rank tensor from the fourth rank tensor  $R_{kl}^{ij}$ . (2016)
- 12. If  $A^{ij}$  and  $B_{mn}$  are tensors, show that two inner products of rank 2 and 0 respectively can be formed.
- 13. A covariant tensor of rank 2 and a contravariant tensor of rank 1 are given by  $g_{\mu\nu}$  and  $A^{\alpha}$  respectively. How can you obtain a tensor of rank 1 by combining them? (2016)
- 14. Find inner product of two tensors  $A^i$  and  $B_j$ .

(Hint:  $j = i \rightarrow A^i B_i = \sum_{i=1}^N A^i B_i = A^1 B_1 + A^2 B_2 + \dots + A^N B_N$ , in *N* dimensional coordinate system )

- 15. If  $R_{ij}$  and  $g^{ij}$  are two tensors, what is the rank of the quantity  $g^{ij}R_{ij}$ ? (2014)
- 16. What is the rank of  $A^i B_i$ ? (2014)
- 17. What is the rank of the quantity  $A^{ijkl}B_{iikp}$ ? (2015)
- 18. What do you mean by an invariant? (2015)
- 19. Find the inner product of two tensors  $A^i$  and  $B_j$ . Hints: Outer product  $A^i X B_j = C_j^i$ Contraction j=i

Inner product  $C_i^i = A^i \times B_i = \sum_i A^i B_i = A^1 B_1 + A^2 B_2 + \cdots$ , no free index, resulting sum is a zero rank tensor (scalar) and similar to the scalar or dot product of two vectors.

- 20. Show that the contraction of outer product of tensors  $C^m$  and  $D_q$  is invariant. (2018)
- 21. If  $A^i$  and  $B_i$  are contravariant and covariant tensors respectively, show that  $A^iB_i$  is invariant under coordinate transformation  $x^i \to x'^i$ . (2015) (Hint: Start with  $\overline{A}^p \overline{B}_q$ , apply transformation rules, then apply contraction q = p, use the property of Kronecker Delta, finally you get  $\overline{A}^p \overline{B}_p = A^i B_i$ )

- 22. If  $g_{ij}$  is a second rank covariant tensor and  $A^i$  and  $B^j$  are two contravariant tensor of rank 1, show that the quantity  $g_{ij}A^iB^j$  remains invariant under coordinate transformation  $x^i \rightarrow x'^i$ . (2016) (Hint: Start with  $\overline{g}_{pq}\overline{A}^r\overline{B}^s$ , apply transformation rules, then apply contractions r = pand s = q, use the property of Kronecker Delta, finally you get  $\overline{g}_{pq}\overline{A}^p\overline{B}^q = g_{ij}A^iB^j$ )
- 23. If  $\frac{\partial x^p}{\partial x^q} = \delta_q^p$  and  $A_j$  is a covariant tensor of rank 1, show that  $C^i = B^{ij}A_j$  transforms like a contravariant tensor of rank 1 under coordinate transformation  $x^i \to x'^i$ . Here  $B^{ij}$  is an arbitrary contravariant tensor of rank 2. (2015) (Hint: Start with  $\overline{B}^{pq}\overline{A}_r$ , apply transformation rules, then apply contraction r = q, use the property of Kronecker Delta, finally you get  $\overline{B}^{pq}\overline{A}_q = \left(\frac{\partial \overline{x}^p}{\partial x^i}\right)B^{ij}A_j = \left(\frac{\partial \overline{x}^p}{\partial x^i}\right)C^i$ )
- 24. Apply contraction on  $\delta_j^i$  and find its value.

(Hint: 
$$j = i \rightarrow \delta_i^i = \sum_{i=1}^N \delta_i^i = \delta_1^1 + \delta_2^2 + \dots + \delta_N^N = 1 + 1 + \dots + 1 = N$$
, in N dimensional coordinate system.)

- 25. If  $I^{ij}$  is a contravariant tensor of rank 2 and  $\omega_j$  is a covariant tensor of rank 1, show that  $L^i = I^{ij}\omega_j$  transforms like a contravariant tensor of rank 1. (2016)
- 26. If  $A_k^{ij}$  and  $B_q^p$  are tensors, show that  $A_k^{ij}B_q^i$  is not a tensor. (2017) Hints:

$$\begin{aligned}
\bar{A}^{\alpha\beta}_{\gamma}\bar{B}^{\rho}_{\sigma} &= \left(\frac{\partial\bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial\bar{x}^{\beta}}{\partial x^{j}}\frac{\partial x^{k}}{\partial\bar{x}^{\gamma}}A^{ij}_{k}\right)\left(\frac{\partial\bar{x}^{\rho}}{\partial x^{p}}\frac{\partial x^{q}}{\partial\bar{x}^{\sigma}}B^{p}_{q}\right)\\
\bar{A}^{\alpha\beta}_{\gamma}\bar{B}^{\alpha}_{\sigma} &= \left(\frac{\partial\bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial\bar{x}^{\beta}}{\partial x^{j}}\frac{\partial x^{k}}{\partial\bar{x}^{\gamma}}A^{ij}_{k}\right)\left(\frac{\partial\bar{x}^{\alpha}}{\partial x^{p}}\frac{\partial x^{q}}{\partial\bar{x}^{\sigma}}B^{p}_{q}\right)
\end{aligned}$$

The RHS of the above expression could not reproduce a tensorial transformation rule for  $A_k^{ij} B_a^i$ , so  $A_k^{ij} B_a^i$  is not a tensor.

27. Express in proper tensor form

$$A(p,q,r)B_r^q + G(t,p)D_t = E(t,s,k)F_{sk}^{tp}$$

(Hint: Apply the rules of addition and inner product, e.g.  $E(t, s, k) F_{sk}^{tp} \rightarrow E_t^{sk} F_{sk}^{tp} = \alpha^p$ .) The two terms in LHS must be 1<sup>st</sup> rank contravariant tensor with contravariant index p.)

28. If  $A_{km}^{ijp}$  is a tensor, show that  $A_{km}^{kmp}$  is a contravariant vector. (2018) Hints: Transformation rule

$$\bar{A}^{\alpha\beta\gamma}_{\rho\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{p}} \frac{\partial x^{k}}{\partial \bar{x}^{\rho}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ijp}_{km}$$

Applying contraction  $\alpha = \rho$ ,  $\beta = \sigma$ 

$$\begin{split} \bar{A}^{\rho\sigma\gamma}_{\rho\sigma} &= \frac{\partial \bar{x}^{\rho}}{\partial x^{i}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{p}} \frac{\partial x^{k}}{\partial \bar{x}^{\rho}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ijp}_{km} \\ &= \frac{\partial x^{k}}{\partial x^{i}} \frac{\partial x^{m}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{p}} A^{ijp}_{km} \\ &= \delta^{k}_{i} \delta^{m}_{j} \frac{\partial \bar{x}^{\gamma}}{\partial x^{p}} A^{ijp}_{km} \\ &\bar{A}^{\delta\sigma\gamma}_{\delta\sigma} &= \frac{\partial \bar{x}^{\gamma}}{\partial x^{p}} A^{kmp}_{km}, \end{split}$$

which is exactly similar to the transformation rule followed by a contravariant vector (contravariant 1<sup>st</sup> rank tensor). Thus,  $A_{km}^{kmp}$  is a contravariant vector.

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