ELECTROMAGNETIC THEORY

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EM Waves & Maxwell's Field Equations

1.1 Introduction

Perhaps the greatest theoretical achievement of physics in the 19th century was the discovery of electromagnetic waves. The history of electromagnetic theory begins with ancient measures to understand atmospheric electricity, in particular lightning, but were unable to explain the phenomena. People then had little understanding of electricity except these facts. Electric forces in nature come in two kinds. First, there is the electric attraction between unlike (+) and (−) charges or repulsion between like (+) and (+) or (−) and (−) electric charges. It is possible to use this to define a unit of electric charge, as the charge which repels a similar charge at a distance of, say, 1 meter, with a force of unit strength.

Then Faraday showed that a magnetic field which varied in time like the one produced by an alternating current (AC) could drive electric currents, if (say) copper wires were placed in it in the appropriate way. That was "magnetic induction," the phenomenon on which electric transformers are based. So, magnetic fields could produce electric currents, and we already know that electric currents produce magnetic fields. In the 19th century it had become clear that electricity and magnetism were related, and their theories were unified: wherever charges are in motion electric current results, and magnetism is due to electric current. The source for electric field is electric charge, whereas that for magnetic field is electric current (charges in motion).

In 1864 Maxwell theoritically proposed that electromagnetic disturbance travels in free space with the speed of light. Although the idea was remain hidden in his set of equations but virtually never said anything about the waves nor he said anything about the generation of such waves. Later on

• Hertz in 1888 succeeded in producing and observing electromagnetic waves of wavelength of the order of 6m in the laboratory.

• J. C. Bose in 1895 succeeded in producing and observing electromagnetic waves of much shorter wavelength 25 mm - 5 mm.

• G. Marconi in the same year succeeded in transmitting electromagnetic waves over distances of many kilometers.

1.2 Basics & Terminologies

Whereas the Lorentz force law characterizes the observable effects of electric and magnetic fields on charges, Maxwells equations characterize the origins of those fields and their relationships to each other. The simplest representation of Maxwells equations is in differential form, which leads directly to waves. But before going to the Maxwell's equations let us refresh ourself with few terminologies which will have regular appearances in the equations. The four Maxwell

Field variables	Names	Unit
$\vec{E}$	Electric Field	volts/meter; Vm^{-1}
$\vec{H}$	Magnetic Intensity	amperes/meter; Am^{-1}
$\vec{B}$	Magnetic Flux Density	Tesla, T
$\vec{D}$	Electric Displacment	coulombs/m^2; Cm^{-2}
$\vec{J}$	Electric current density	amperes/m^2; Am^{-2}
ρ	Electric charge density	coulombs/m^3; Cm^{-3}

equations which will be derived in the next section invoke one scalar and five vector quantities comprising 16 variables. Some variables only characterize how matter alters field behavior. In vacuum we can eliminate three vectors (9 variables) by noting $\vec{D} = \epsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$ and $\vec{J} = \rho \vec{v} = \sigma \vec{E}$. where $\epsilon_0 = 8.8542 \times 10^{-12}$ [farads m^{-1}] is the **absolute permittivity of vacuum**, $\mu_0 = 4\pi \times 10^{-7}$ [henries m^{-1}] is the **absolute permeability of vacuum**, $\vec{v}$ is the drift velocity of the local net charge density ρ, and σ is the conductivity of a medium [Siemens m^{-1}]. If we regard the electrical sources ρ and $\vec{J}$ as given, then the equations can be solved for all remaining unknowns. Specifically, we can then find $\vec{E}$ and $\vec{B}$, and thus compute the forces on all charges present.

1.3 Basic vector opearation

Before deriving the equations we will look at some of the needed vector analysis. We will review the basic vector operations (the dot and cross products), define the gradient, but mainly curl, and divergence that are often seen in this part of physics courses. Equipped with these vector operations, we will derive the three dimensional waves equation for electromagnetic waves from Maxwells equations. Let us quickly look it for an electric field given by $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$.

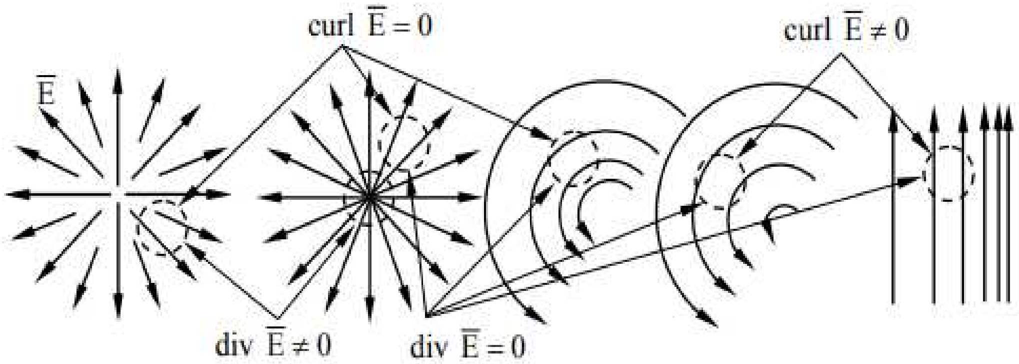
Curl of $\vec{E}$

$$\vec{\nabla} \times \vec{E} = \left(\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}\right) \times (E_x\hat{i} + E_y\hat{j} + E_z\hat{k})$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ E_x & E_y & E_z \end{vmatrix}$$

Divergence of $\vec{E}$

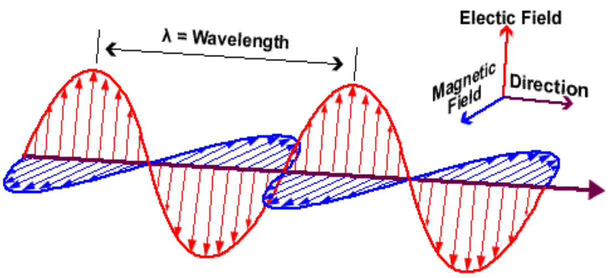
$$\vec{\nabla} \bullet \vec{E} = \left(\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}\right) \bullet (E_x\hat{i} + E_y\hat{j} + E_z\hat{k})$$
$$= \frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_x}{dz}$$

The next figure illustrates when the divergence and curl are zero or non-zero for five representative field distributions.



1.3.1 E -M Waves

• **Definition:**
Electromagnetic waves or EM waves are oscillating magnetic and electric fields at right angles to each other, self-propagating in direction perpendicular to both the electric and magnetic fields.



1.3.2 Properties of E -M Waves

Listed below are some of the important characteristics of electromagnetic waves.

• **Property 1:** In electromagnetic waves the electric field vector $\vec{E}$ and magnetic field vector $\vec{B}$ and propagation vector $\vec{K}$ are mutually perpendicular for a right handed system. Hence electromagnetic waves are transverse in nature.
• **Property 2:** Electromagnetic waves travel with speed of light.
• **Property 3:** Electromagnetic waves are self-propagating. They keep on moving even without the source that created them.
• **Property 4:** Electromagnetic waves doesn't need any medium to propagate.

• **Property 5:** Electromagnetic waves are not deflected by electric or magnetic field.
• **Property 6:** Electromagnetic waves can show interference or diffraction and can be polarized.
• **Property 7:** Electromagnetic waves carry energy with them and exerts pressure on the medium they incident upon.

1.4 Maxwell's Field Equations

From a long view of the history of mankind seen from, say, ten thousand years from now, there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics.
.
The Feynman Lectures on Physics (1964), Richard Feynman

Maxwell's equations are a set of four differential equations that, together with the Lorentz force law, form the foundation of classical electromagnetism, classical optics, and electric circuits. These equations describe how electric and magnetic fields propagate, interact, and how they are influenced by objects. He was an Einstein/Newton-level genius who took a set of known experimental laws (Faraday's Law, Ampere's Law) and unified them into a symmetric coherent set of Equations known as Maxwell's Equations. Maxwell was one of the first to determine the speed of propagation of electromagnetic (EM) waves was the same as the speed of light - and hence to conclude that EM waves and visible light were really the same thing.
The four Maxwell's equations can be divided into two major subsets. The first two, Gauss's law for electrostatics and one people used to say as Gauss's law for magnetism, however it is not exactly so, describe how fields emanate from charges and magnets respectively. The other two, Faradays law and Ampere's law with Maxwell's correction, describe how induced electric and magnetic fields circulate around their respective sources.
Each of Maxwell's equations can be looked at from the "microscopic" perspective, which deals with total charge and total current, and the "macroscopic" set, which defines two new auxiliary fields that allow one to perform calculations without knowing microscopic data like charges at the atomic level.

Let us now discuss them one by one.

1.4.1 Maxwell's equation, Gauss Law for electrostatics

The integral of the outgoing electric field $\vec{E}$ over an area enclosing a volume V equals the total charge Q enclosed by the volume divided by ϵ_0 in vacuum. Mathematically Gauss' law is

$$\oint\!\!\oint_S \vec{E} \bullet d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

But this can be written as an equality between three dimensional volume integrals, by writing the total charge enclosed Q as the integral of the charge density over the volume, using the Gauss' divergence theorem (in fact it is due to him)

$$\begin{aligned}
\oint\!\!\oint_S \vec{E} \bullet d\vec{S} &= \frac{1}{\epsilon_0} \iiint_V \rho \, dV \\
\iiint_V (\vec{\nabla} \bullet \vec{E}) \, dV &= \frac{1}{\epsilon_0} \iiint_V \rho \, dV \\
\boxed{\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\epsilon_0}}
\end{aligned}$$

Where ρ is the volume charge density.
It represents completely covering the surface with a large number of tiny patches having areas $d\vec{S}$. We represent these small areas as vectors pointing outwards, because we can then take the dot product with the electric field to select the component of that field pointing perpendicularly outwards (it would count negatively if the field were pointing inwards) - this is the only component of the field that contributes to actual flow across the surface. (Just as a river flowing parallel to its banks has no flow across the banks).

• **Physical Significance:**
The net quantity of the electric flux leaving a volume is proportional to the charge inside the volume.

1.4.2 Maxwell's second equation

The second law states that there are no "magnetic charges (or monopoles)" analogous to electric charges, and that magnetic fields are instead generated by magnetic dipoles. Such dipoles can be represented as loops of current, but in many ways are similar in appearance to positive and negative "magnetic charges" that are inseparable and thus have no formal net "magnetic charge." This can be derived from Biot-Savart Law

If $B(r)$ is the magnetic flux at the point r and $J(r)$ is the current density at the point r' then Biot-Savart Law is given by

$$B\,(r) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(r)\,dv\,\times\,\hat{r}}{\left|r\,-\,r'\right|^2}$$

$$\vec{\nabla}\bullet B\,(r) = \frac{\mu_0}{4\pi}\int_V \vec{\nabla}\bullet\frac{\vec{J}(r)\,dv\,\times\,\hat{r}}{\left|r\,-\,r'\right|^2}$$

To carry through the divergence of the integrand in the above equation, we will use the vector identity given by $\vec{\nabla}\bullet(A\,\times\,B) = \vec{B}\bullet(\nabla\,\times\,B) - \vec{A}\bullet(\nabla\,\times\,B)$

$$\vec{\nabla}\bullet B\,(r) = \frac{\mu_0}{4\pi}\int_V\left[\vec{J}(r)\bullet\left(\nabla\,\times\,\frac{\hat{r}}{\left|r\,-\,r'\right|^2}\right)\right]dv - \int_V\left[\frac{\hat{r}}{\left|r\,-\,r'\right|^2}\bullet\left(\nabla\,\times\,\vec{J}(r)\right)\right]dv$$

The first part of RHS of the above equation is zero as the curl of $\dfrac{\hat{r}}{\left|r\,-\,r'\right|^2}$ is zero. Also the second part of RHS of the above equation becomes zero because $J(r)$ depends on r' and ∇ depends only on r. Plugging this back into, the right-hand side of the expression becomes zero. Thus, we see that

$$\vec{\nabla}\bullet\vec{B} = 0$$

Another way of doing the same thing is the following

$$\oiint_S \vec{B}\bullet d\vec{S}\; = \;0$$

Now applying the Gauss' divergence theorem to the above equation we get

$$\iiint_V (\vec{\nabla}\bullet\vec{B})\,dV\; = \;0$$

$$\boxed{\vec{\nabla}\bullet\vec{B} = 0}$$

Magnetic field lines form loops such that all field lines that go into an object leave it at some point. Thus, the total magnetic flux through a surface surrounding a magnetic dipole is always zero.

• **Physical Significance:**
Magnetic monopole doesnot exsist.

1.4.3 Maxwell's third equation, Faraday's law of EM induction

Faraday demonstrated the fact that whenever the magnetic flux associated with any closed loop changes an induced emf developes in the circuit and that sends current through the circuit which last so long as the change of flux lasts. He also showed that the induced emf produced is directly proporsonal to magnetic flux linked with the coil. In mathematical language Faradays law states that the closed integral of the induced electric field is minus the time rate of change of the magnetic flux through the loop. Or simply saying a time-varying magnetic field (or flux) induces an electric field. In fact the straight forward outcome of this equation says that work is needed to take a charge around a closed curve in an electric field.
Thus the mathematical form is

$$\mathcal{E}\; = \;-\frac{d\Phi_B}{dt}$$

$$\oint_C \vec{E}\bullet d\vec{l}\; = \;-\frac{\partial\Phi_B}{\partial t}$$

$$\oint_C \vec{E}\bullet d\vec{l}\; = \;-\iint_S \frac{\partial\vec{B}}{\partial t}\bullet d\vec{S}$$

where $\mathcal{E}$ is the induced emf. It may seem that the integral on the right hand side is not very clearly defined, because if the path or circuit lies in a plane, the natural choice of spanning surface is flat, but how do you decide what surface to choose to do the integral over for a wire bent into a circuit that doesn't lie in a plane? The answer is that it doesn't matter what surface you choose, as long as the wire forms its boundary. Now applying Stokes' theoram in the above equation we get

$$\oiint_S(\vec{\nabla}\,\times\,\vec{E})\bullet d\vec{S}\; = \;-\oiint_S \frac{\partial\,\vec{B}}{\partial\,t}\bullet d\vec{S}$$

$$\boxed{\vec{\nabla}\,\times\,\vec{E} = -\frac{\partial\,\vec{B}}{\partial\,t}}$$

The divergence of the left hand side of Faraday's law, $\vec{\nabla}\bullet(\vec{\nabla} \times \vec{E})$, vanishes identically so if Faradays law is consistent it must be true that $\vec{\nabla}\bullet\frac{\partial B}{\partial t}$ also vanishes. Since the time and space partial derivatives commute, this is the same as $\frac{d}{dt}\ \vec{\nabla}\bullet B$, which vanishes thanks to the second law. So the absense of magnetic charges is required for Faraday's law to be self-consistent.

• **Physical Significance:**
A time varying magnetic field linked with a loop produces an induced emf in the loop which in turn produces a space varying electric field.

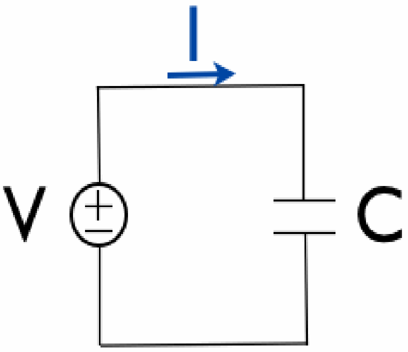
1.4.4 Maxwell's fourth equation, Modification of Ampere's law

Amperes law states that the line integral of the magnetic field $\vec{B}$ around any closed path or circuit is equal to the current enclosed by the path. In a simple note magnetic field could be created by electrical current. ie

$$\oint_C \vec{B}\bullet\vec{dl} = \mu_0 I$$

The I in Ampere's law is called the conduction current, ie $\frac{dQ}{dt}$. But this Ampere's law is in its incomplete form. Why? Because from here we can show that $\vec{\nabla} \times \vec{B} = \mu_0\,\vec{J}$. Now taking divergence in this equation the left hand side vanishes. ie $\vec{\nabla}\bullet\vec{\nabla} \times \vec{B} = 0$. But the at the same time the right hand side doesn't go off since according to equation of continuity $\frac{\partial\rho}{\partial t} + \vec{\nabla}\bullet\vec{J} = 0$ which gives $\vec{\nabla}\bullet\vec{J} = -\frac{d\rho}{dt}$. In a steady state situation, where all time derivatives vanish, Ampre's law is self-consistent. However in the presence of time dependent charge densities it cannot be correct. Because here electric field which grows continuously since there has been an accumulation of charge in the capacitor plates. Thus there is a time varying electric field present between the plates. That implies there must also a magnetic field present inside the capacitor plate. And then if you place a compass needle between the capacitor plates the needle gets displaced ie there is some deflection. So the point here is that between the plates no conductor is there ie no conduction current should be there but still the needle is showing deflection and the circuit shows a current reading. Maxwell resolved this contradiction by creating something called a displacement current. This was an analogy with a dielectric material. If a dielectric material is placed in an electric field, the molecules are distorted, their positive charges moving slightly to the right, say, the negative charges slightly to the left. Now consider what happens to a dielectric in an increasing electric field. The positive charges will be displaced to the right by a continuously increasing distance, so, as long as the electric field is increasing in strength, these charges are moving: there is actually a displacement current. This electric field that produces the current and makes the circuit continuous. Maxwell added this displacement term in Ampere's law and he showed that it is equal to the permittivity of free space times the rate of change of electric flux with respect to time. Let us now see how this was achieved.

$$\begin{aligned}
C &= \frac{Q}{V}\\
\frac{\epsilon_0\,A}{d} &= \frac{Q}{V}\\
Q &= \epsilon_0\,A\,\frac{V}{d}\\
\frac{dQ}{dt} &= \epsilon_0\,A\,\frac{d\vec{E}}{dt}\\
I_D &= \epsilon_0\,A\,\frac{d\vec{E}}{dt}
\end{aligned}$$



where I_D is the displacement current and other symbols have their usual meaning. Thus Maxwell modified the Amperes law to

$$\oint_C \vec{B}\bullet\vec{dl} = \mu_0\,(I + I_D)$$

Now the above equation can be rewritten as

$$\begin{aligned}
\oint_C \vec{B}\bullet\vec{dl} &= \mu_0\left[\oiint_S \vec{J}\bullet\vec{dS} + \oiint_S \epsilon_0\,\frac{d\vec{E}}{dt}\bullet\vec{dS}\right]\\
\iint_S (\vec{\nabla} \times \vec{B})\bullet\vec{dS} &= \mu_0\left[\iint_S \vec{J}\bullet\vec{dS} + \iint_S \epsilon_0\,\frac{d\vec{E}}{dt}\bullet\vec{dS}\right]
\end{aligned}$$

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0\left[\vec{J} + \epsilon_0\,\frac{d\vec{E}}{dt}\right]}$$

Therefore, this is the way to generalize Ampere's law from the magnetostatic situation to the case where charge densities are varying with time.

• **Physical Significance:**
Magnetic field B around any closed path or circuit is equal to the conductions current plus the time derivative of electric displacement through any surface bounded by the path.

1.5 Maxwell's four equations:

Equation No.	Equation	Remark
1	$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\epsilon_0}$	Gauss law for electrostatics
2	$\vec{\nabla} \bullet \vec{B} = 0$	No name
3	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Faraday's law of Electro-magnetic induction
4	$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$	Maxwell's modification of Ampere's law

1.6 Explicit solutions of Maxwell's equations

As we have come to know that these celebrated Maxwell's equations are responsible for any of electro-magnetic phenomenon in fact for every electro-magnetic phenomenon. So it is much desired to have the solutions for $\vec{E}$ and $\vec{B}$ fields. Using the principles of vector algebra we find that

$$\vec{\nabla} \bullet \vec{B} = 0$$
$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

Here $\vec{A}$ is the magnetic vector potential which is not directly associated with work the way that scalar potential is. One rationale for the vector potential is that it may be easier to calculate the vector potential than to calculate the magnetic field directly from a given source current geometry. (you don't need to think about it very much! Just remember it). Now putting this value of $\vec{B}$ in Maxwell's 3rd equation we get

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$$
$$\vec{\nabla} \times \left[\vec{E} + \frac{\partial \vec{A}}{\partial t}\right] = 0$$
$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\Phi$$
$$\boxed{\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}}$$

Where Φ is the scalar potential. Thus because of a changing magnetic field the curl of the electric field becomes non-zero and we donot need to abandon $\vec{E}$ is not a conservative field. The last equation tells us that the scalar potential Φ only describes the conservative electric field generated by electric charges. The electric field induced by time-varying magnetic fields is non-conservative, and is described by the magnetic vector potential $\vec{A}$.

Chapter 2

Propagation of Electro-Magnetic Waves

2.1 Medium Characteristics

When we consider a medium which is "simple", we define it by the following characteristics
• **Linear Medium**: Here μ and ϵ are constants. In general these two are tensorial terms.

• **Isotropic Medium**: Here the EM wave travels at same speed in all directions. ie there is no special direction is preferred which further implies that rotational symetry is present.

• **Homogeneous Medium**: By this we mean that the material is uniform. ie to speak that the only one single material is present in the medium which implies that the density is fixed at every point in space. Thus translation symetry is present.

• **Source-free Medium**: By this we mean that charge density ie $\rho = 0$.

• **Non-conducting Medium**: Here the conductivity $\sigma = 0$. Hence current density $\vec{J} = \sigma \vec{E} = 0$.

2.2 Propagation of EM waves in Free Space ie $\sigma = 0$, & $\rho = 0$

An electromagnetic wave transports its energy through a vacuum at a speed of 3×10^8 m/sec. However, Mechanical waves, unlike electromagnetic waves, require the presence of a material medium in order to transport their energy from one location to another. Following is the way by which we can show that EM wave indeed travels at the speed of light.
Let us do it for the $\vec{E}$. Starting with Maxwell's third equation

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

Now taking curl on both sides we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla}(\vec{\nabla} \bullet \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Now replacing $\vec{\nabla} \times \vec{B}$ by Maxwell's 4th equation and $\vec{\nabla} \bullet \vec{E}$ by Maxwell's 1st equation we get

$$\vec{\nabla} \frac{\rho}{\epsilon_0} - \nabla^2 \vec{E} = -\mu_0 \frac{d}{dt}\left[\vec{J} + \epsilon_0 \frac{d\vec{E}}{dt}\right]$$
$$\vec{\nabla} \frac{\rho}{\epsilon_0} - \nabla^2 \vec{E} = -\mu_0 \frac{d}{dt}\left[\sigma\vec{E} + \epsilon_0 \frac{d\vec{E}}{dt}\right]$$
$$\vec{\nabla} \frac{\rho}{\epsilon_0} - \nabla^2 \vec{E} = -\mu_0 \sigma \frac{\partial \vec{E}}{\partial t} - \mu_0 \epsilon_0 \frac{d^2\vec{E}}{dt^2}$$
$$\nabla^2 \vec{E} - \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{\nabla} \frac{\rho}{\epsilon_0}$$

In free space no charge accumulation, nothing is there hence $\rho = 0$ and also free space conducts nothing means $\sigma = 0$. Under these two situation the above expression boils down to

$$\nabla^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$
$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Thus we have a 2nd order differential equation where the 2nd order space derivative of function is proporsonal to the 2nd order time derivative of the same function. So that's your wave equation. And the reciprocal of the proporsonality constant gives the square of velocity of propagation of the function.

$$v^2 = \frac{1}{\mu_0 \epsilon_0}$$

Now plugging in the value for $\mu_0 = 4\pi \times 10^{-7}$H/m and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m in the above equation and simplifying we get

$$\boxed{v = 3 \times 10^8 \text{ m/sec} = \text{speed of light, c}}$$

Since the value of the speed of EM-wave is similar to that the speed of light therefore a corelation can be drawn that light is a form of EM-wave. OR

There is another way to get to the same result. The equations are now decoupled (E has its own private equations), which certainly simplifies things, but in the process we've changed them from first to second order (notice all the squares). I know that lower order implies easier to work with, but these second order equations aren't as difficult as they look. Raising the order has not made things more complicated, it's made things more interesting. What we will assume is the following, a plane wave solution for the $\vec{E}$. ie

$$\vec{E} = \vec{E_0}\, e^{i(\omega t - \kappa z)}$$

Therefore $\nabla^2 \vec{E} = -\kappa^2 \vec{E}$ and $\frac{d^2 \vec{E}}{dt^2} = -\omega^2 \vec{E}$. And substituting these values in the last differentail equation we get

$$-\kappa^2 \vec{E} = -\mu_0 \epsilon_0 \omega^2 \vec{E}$$
$$\kappa^2 = \mu_0 \epsilon_0 \omega^2$$
$$\frac{\omega^2}{\kappa^2} = \frac{1}{\mu_0 \epsilon_0}$$
$$v^2 = \frac{1}{\mu_0 \epsilon_0}$$

Now plugging in the value for $\mu_0 = 4\pi \times 10^{-7}$H/m and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m in the above equation and simplifying we get

$$\boxed{v = 3 \times 10^8 \text{ m/sec} = \text{speed of light, c}}$$

So similar type of results can also be obtained via this method. Similarly we can also show that this is true also for $\vec{B}$. All you have to do is to start with Maxwell's 4th equation and similar type of vectorial algebra.

2.3 Impedance of free space

The characteristic impedance of free space, also called the Z_0 of free space, is an expression of the relationship between the electric-field and magnetic-field intensities in an electromagnetic field (EM field) propagating through a vacuum, the analogous quantity for a plane wave travelling through a dielectric medium is called the intrinsic impedance of the medium. The Z_0 of free space, like characteristic impedance in general, is expressed in ohms, and is theoretically independent of wavelength. It is considered a physical constant. However, with the redefinition of the SI base units which has been already gone into force on May 20, 1919, this value is subject to experimental measurement. Let us now derive the of this impedance.
Let us suppose that an EM wave which is propagating along z direction has $\vec{E}$ along x direction and $\vec{B}$ along y direction. From the last section we now know that electric and magnetic field vector satisfy the wave equation ie second order space derivative is proporsonal to the second order time derivative ie $\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$ and likewise for $\vec{B}$ field also. And this has a plane wave solution as $\vec{E} = \vec{E_0}\, e^{i(\omega t - \kappa z)}$.

Now using Maxwell's 4th equation we get

$$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{d\vec{E}}{dt}\right] = \mu_0 \left[\sigma \vec{E} + \epsilon_0 \frac{d\vec{E}}{dt}\right]$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \qquad \text{(for free space } \sigma = 0\text{)}$$

$$\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{B_x} & \vec{B_y} & \vec{B_z} \end{bmatrix} = \mu_0 \epsilon_0 \left[\frac{\partial \vec{E_x}}{\partial t} + \frac{\partial \vec{E_y}}{\partial t} + \frac{\partial \vec{E_z}}{\partial t}\right]$$

$$\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \vec{B_y} & 0 \end{bmatrix} = \mu_0 \epsilon_0 \left[\frac{\partial \vec{E_x}}{dt}\right] \qquad (\vec{E} \text{ is along x, } \vec{B} \text{ is along y})$$

$$-\frac{\partial B_y}{\partial z} = \mu_0 \epsilon_0 \frac{\partial \vec{E_x}}{\partial t}$$

$$i \kappa B = \mu_0 \epsilon_0 i \omega E$$

$$i \kappa H = \epsilon_0 i \omega E$$

$$\frac{E}{H} = \frac{\kappa}{\epsilon_0 \omega} = \frac{1}{\epsilon_0 c} = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

Now plugging in the value for $\mu_0 = 4\pi \times 10^{-7}$H/m and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m in the above equation and simplifying we get

$$\boxed{\frac{E}{H} = 376.6 \quad \text{ohms}}$$

Mathematically, the Z_0 of free space is equal to the square root of the ratio of the permeability of free space in henrys per meter to the permittivity of free space in farads per meter. The Z_0 of dry air is similar to that of free space, because dry air has little effect on permeability or permittivity. However, in environments where the air contains seawater spray, excessive humidity, heavy precipitation, or high concentrations of particulate matter, the Z_0 is slightly reduced.

2.4 Propagation of EM waves in Conducting Medium ie, $\sigma \neq 0$

The mechanism of propagation of EM waves in a medium along with the energy transport through a medium involves the absorption and reemission of the wave energy by the atoms of the material. When an electromagnetic wave impinges upon the atoms of a material, the energy of that wave is absorbed. The absorption of energy causes the electrons within the atoms to undergo vibrations. After a short period of vibrational motion, the vibrating electrons create a new electromagnetic wave with the same frequency as the first electromagnetic wave. While these vibrations occur for only a very short time, they delay the motion of the wave through the medium. Once the energy of the electromagnetic wave is reemitted by an atom, it travels through a small region of space between atoms. Once it reaches the next atom, the electromagnetic wave is absorbed, transformed into electron vibrations and then reemitted as an electromagnetic wave.
The actual speed of an electromagnetic wave through a material medium is dependent upon the optical density of that medium. Different materials cause a different amount of delay due to the absorption and reemission process. Furthermore, different materials have their atoms more closely packed and thus the amount of distance between atoms is less. These two factors are dependent upon the nature of the material through which the electromagnetic wave is traveling. As a result, the speed of an electromagnetic wave is dependent upon the material through which it is traveling.
Starting with Maxwell's third equation

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

Now taking curl on both sides we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}(\vec{\nabla} \bullet \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Now replacing $\vec{\nabla} \times \vec{B}$ by Maxwell's 4th equation and $\vec{\nabla} \bullet \vec{E}$ by Maxwell's 1st equation we get

$$\vec{\nabla}\frac{\rho}{\epsilon} - \nabla^2\vec{E} = -\mu\,\frac{\partial}{\partial\,t}\left[\vec{J} + \epsilon\,\frac{d\vec{E}}{dt}\right]$$

$$\vec{\nabla}\frac{\rho}{\epsilon} - \nabla^2\vec{E} = -\mu\,\frac{\partial}{\partial\,t}\left[\sigma\vec{\vec{E}} + \epsilon\,\frac{\partial\,\vec{E}}{\partial\,t}\right]$$

$$\vec{\nabla}\frac{\rho}{\epsilon} - \nabla^2\vec{E} = -\mu\,\sigma\,\frac{\partial\,\vec{E}}{\partial\,t} - \mu\,\epsilon_0\,\frac{\partial^2\vec{E}}{\partial\,t^2}$$

$$\nabla^2\vec{E} - \mu\,\sigma\,\frac{\partial\,\vec{E}}{\partial\,t} - \mu\,\epsilon\frac{\partial^2\vec{E}}{\partial\,t^2} = \vec{\nabla}\frac{\rho}{\epsilon}$$

In no charge accumulation nothing is there then $\rho = 0$. In such cases the last expression will take the shape

$$\nabla^2\vec{E} - \mu\,\sigma\,\frac{\partial\,\vec{E}}{\partial\,t} - \mu\,\epsilon\frac{\partial^2\vec{E}}{\partial\,t^2} = 0$$

Assuming a plane wave solution for the $\vec{E}$. ie

$$\vec{E} = \vec{E_0}\,e^{-i\,(\omega t\,-\,\kappa z)}$$

Therefore $\nabla^2\vec{E} = -\kappa^2\vec{E}$, $\frac{\partial\,\vec{E}}{\partial\,t} = -i\omega\vec{E}$ and $\frac{\partial^2\vec{E}}{\partial\,t^2} = -\omega^2\vec{E}$. And substituting these values in the last differentail equation we get

$$-\kappa^2\vec{E} = -i\sigma\,\mu\,\omega\vec{E} - \mu\,\epsilon\,\omega^2\vec{E}$$

$$-\kappa^2 = -i\sigma\,\mu\,\omega - \mu\,\epsilon\,\omega^2$$

$$\kappa^2 = \mu\,\epsilon\,\omega^2 + i\sigma\,\mu\,\omega$$

$$\kappa^2 = \mu\,\epsilon\,\omega^2\left[1 + i\,\frac{\sigma}{\epsilon\,\omega}\right]$$

Thus it is clear that the square of the propagation constant is a complex quantity. Hence it is quiet legitimate to assume the propagation constant as an other complex quantity and then to equating it so that we have something meaningful. So in this notion let us assume

$$\kappa = \alpha + i\,\beta$$

$$\kappa^2 = (\alpha + i\,\beta)^2$$

$$\mu\,\epsilon\,\omega^2\left[1 + i\,\frac{\sigma}{\epsilon\,\omega}\right] = \alpha^2 - \beta^2 + i\,2\,\alpha\,\beta$$

Since we know the fact that for two complex numbers to be equal, then the real parts must be equal and the imaginary parts must be equal. So one equation involving complex numbers can be written as two equations, one for the real parts, one for the imaginary parts.

$$\alpha^2 - \beta^2 = \mu\,\epsilon\,\omega^2 \qquad \text{and} \qquad 2\,\alpha\,\beta = (\mu\,\epsilon\,\omega^2)\,\frac{\sigma}{\epsilon\,\omega} = \mu\,\omega\,\sigma$$

$$\beta = \frac{\mu\,\omega\,\sigma}{2\,\alpha}$$

Now solving for α^2 we get

$$\alpha^2 - \left[\frac{\mu\,\omega\,\sigma}{2\,\alpha}\right]^2 = \mu\,\epsilon\,\omega^2$$

$$4\,\alpha^4 - 4\,\alpha^2\mu\,\epsilon\,\omega^2 - (\mu\,\omega\,\sigma)^2 = 0$$

$$\alpha^2 = \frac{-(-4\,\mu\,\epsilon\,\omega^2) \pm \sqrt{(-4\,\mu\,\epsilon\,\omega^2)^2 + 4\times 4\times\mu^2\,\omega^2\,\sigma^2}}{2\times 4}$$

$$= \frac{4\,\mu\,\epsilon\,\omega^2 \pm \sqrt{16\,\mu^2\,\epsilon^2\,\omega^4 + 16\,\mu^2\,\omega^2\,\sigma^2}}{8}$$

$$= \frac{4\,\mu\,\epsilon\,\omega^2 \pm 4\,\mu\,\epsilon\,\omega^2\sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}}{8}$$

$$= \frac{\mu\,\epsilon\,\omega^2}{2}\left[1 \pm \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]$$

$$\alpha = \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[1 \pm \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}}$$

Now putting the value of α^2 in $\alpha^2 - \beta^2 = \mu\,\epsilon\,\omega^2$ and then solving for β^2 we get

$$\begin{aligned}
\beta^2 &= \alpha^2 - \mu\,\epsilon\,\omega^2 \\
&= \frac{\mu\,\epsilon\,\omega^2}{2}\left[1 \pm \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right] - \mu\,\epsilon\,\omega^2 \\
&= \frac{\mu\,\epsilon\,\omega^2}{2}\left[-1 \pm \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right] \\
\beta &= \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[-1 \pm \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}}
\end{aligned}$$

But from both α and β we will drop the "$-$" sign from the $\pm$ since presence of the "$-$" sign doesn't going to give us anything which is physically interpretable. This is why it is. Typical values conductivity of a metal ie the value of σ is $\approx 10^7$ mho/m and ϵ remains almost around 10^{-12} Farad/m. The value of frequency is generally is in the order of Megahertz. So $\frac{\sigma^2}{\epsilon^2\,\omega^2} >> 1$ which will then lead to a complex value for both α and β since analysis will yield $\sqrt{\text{-ve nos.}}$. Thus we have α and β given by

$$\alpha = \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}} \qquad \text{and} \qquad \beta = \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}}$$

Thus the wave solution will be obtained by replacing the value of κ by α and β in the plane wave solution.

$$\begin{aligned}
\vec{E} &= \vec{E_0}\,e^{-i\,(\omega t - \kappa z)} = \vec{E_0}\,e^{-i\,[\omega t - (\alpha + i\,\beta)\,z]} \\
&= \vec{E_0}\,e^{-i\,\omega t + i\,\alpha\,z - i\,\beta\,z} \\
&= \vec{E_0}\,e^{-\,\beta\,z}\,e^{-i\,(\omega t - \alpha\,z})
\end{aligned}$$

Hence the final E field equation (a similar type of B field also) in any medium is given by

$$\boxed{\vec{E} = \vec{E_0}\,exp\left[-\sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}} z\right] exp\left[-i\,(\omega t - \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}} z)\right]}$$

2.4.1 Attenuation

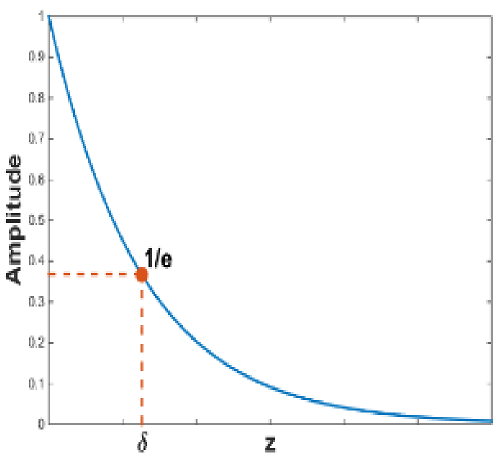
The presence of $e^{-\,\beta\,z}$ term in the equation tells that an EM wave experiences attenuation ie a rate of amplitude loss is present as it propagates through the medium. Attenuation defines the rate of amplitude loss an EM wave experiences at it propagates which is defined by the parameter β. Thus

$$\beta = \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}} > 0$$

2.4.2 Skin Depth

Skin depth defines the distance a wave must travel before its amplitude has decayed by a factor of $\frac{1}{e}$. The skin depth is the reciprocal of the decay constant β. Thus

$$\begin{aligned}
z = \frac{1}{\beta} &= \left[\sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left(-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right)^{\frac{1}{2}}\right]^{-1} \\
z = \frac{1}{\beta} &= \left[\omega\,\sqrt{\frac{\mu\,\epsilon}{2}}\left(-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right)^{\frac{1}{2}}\right]^{-1} \\
&= \left[\omega\,\left(-\frac{\mu\,\epsilon}{2} + \frac{\mu\,\epsilon}{2}\sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right)^{\frac{1}{2}}\right]^{-1}
\end{aligned}$$



But in the quasi-static regime, ie for a good conductor $\frac{\sigma}{\epsilon\,\omega} > 1$. Thus the above expression will take the following form

$$z = \frac{1}{\beta} = \left[\omega\left(-\frac{\mu\,\epsilon}{2} + \frac{\sigma}{\epsilon\,\omega}\frac{\mu\,\epsilon}{2}\right)^{\frac{1}{2}}\right]^{-1} = \left[\omega\left(-\frac{\mu\,\epsilon}{2} + \frac{\mu\,\sigma}{2\,\omega}\right)^{\frac{1}{2}}\right]^{-1} = \left[\left(-\frac{\mu\,\epsilon\,\omega^2}{2} + \frac{\mu\,\sigma\,\omega}{2}\right)^{\frac{1}{2}}\right]^{-1} \approx \sqrt{\frac{2}{\mu\,\sigma\,\omega}}$$

Thus from the last equations, we see that the skin depth decreases as the conductivity σ, magnetic permeability μ and frequency ω increases. In most cases however, the magnetic properties are negligible as $\mu \approx \mu_0$.

2.5 Propagation of EM waves in a Dielectric Medium ie $\sigma = 0$

We now consider electromagnetic waves propagating in a dielectric medium. We suppose that the medium is not magnetized, and we further assume that the waves are propagating in the absence of free charges and currents ie no conduction of charge. Under these assumptions you just have to do all the calculation just we did in case of conducting medium until you find the α and β which are

$$\alpha = \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}} \qquad \text{and} \qquad \beta = \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}}$$

since my conductivity is zero, $\sigma = 0$ the value of α and β will be after putting the value of σ we get

$$\begin{aligned}\alpha &= \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}} \\ &= \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}(1 + 1)^{\frac{1}{2}} \\ &= \sqrt{\mu\,\epsilon}\,\omega\end{aligned} \qquad \begin{aligned}\beta &= \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}\left[-1 + \sqrt{1 + \frac{\sigma^2}{\epsilon^2\,\omega^2}}\right]^{\frac{1}{2}} \\ &= \sqrt{\frac{\mu\,\epsilon\,\omega^2}{2}}(-1 + 1)^{\frac{1}{2}} \\ &= 0\end{aligned}$$

Thus it can be realised that

$$\begin{aligned}\kappa = \alpha + i\,\beta &= \alpha \\ &= \sqrt{\mu\,\epsilon}\,\omega \\ \frac{\omega}{\kappa} &= \frac{1}{\sqrt{\mu\,\epsilon}} = \frac{1}{\sqrt{\mu_r\,\mu_0\,\epsilon_r\,\epsilon_0}} \\ \text{Vel. of the wave, } v &= \frac{c}{\sqrt{\mu_r\,\epsilon_r}}\end{aligned}$$

Thus it is seen that the there will be propagation of the EM wave even in the medium, but the velocity will be less than the speed of light since μ_r & $\epsilon_r > 1$. How much slower? Here is your answer. Let us assume that the medium is non-magnetic material, hence $\mu_r = 1$ and this will result the following

$$\begin{aligned}v &= \frac{c}{\sqrt{\epsilon_r}} \\ \frac{c}{v} &= \sqrt{\epsilon_r} \\ \text{Refractive index of the medium } n &= \sqrt{\epsilon_r}\end{aligned}$$

Hence, we conclude that electromagnetic waves propagate through a dielectric medium slower than through a vacuum by a factor n. This conclusion (which was reached long before Maxwell's equations were invented) is the basis of all geometric optics involving refraction.

2.6 Poynting Theorem

Perhaps the heart of electro-magnetic theory. Let us consider a case where an electromagnetic field confined to a given volume. Now let me ask you a question. How does the energy contained in the field, change? And the answer is actually there are two processes by which it can happen. The first is by the mechanical work done by the electromagnetic field on the currents, which would appear as Joule heat and the second process is by radiative flow of energy across the surface of the volume. Thus in electrodynamics, **Poynting's theorem is a statement of conservation of energy for the electromagnetic field.** It is analogous to the work-energy theorem in classical mechanics, and mathematically similar to the continuity equation, because it relates the energy stored in the electromagnetic field to the work done on a charge distribution (i.e. an electrically charged object), through energy flux.

•The Statement: Version I
The rate of energy transfer (per unit volume) from a region of space equals to the rate of work done which will be stored on a charge distribution plus the energy flux leaving out of that region.
•The Statement: Version II
The decrease in the electromagnetic energy per unit time in a certain volume is equal to the sum of work done by the field forces and the net outward flux per unit time.
•The Statement: Version III
The time rate of change of electromagnetic energy within a volume V plus the net energy flowing out of that volume through a surface S per unit time is equal to the negative of the total work done on the charges within the volume V. You can write whichever you want in your exam. All are equivalent.

2.6.1 Derivation of Poynting Theoram

Consider first a single particle of charge Q traveling with a velocity vector $\vec{v}$. Let $\vec{E}$ and $\vec{B}$ be electric and magnetic fields external to the particle; i.e., $\vec{E}$ and $\vec{B}$ do not include the electric and magnetic fields generated by the moving charged particle. The force on the particle is given by the Lorentz formula

$$F = Q\left[\vec{E} + (\vec{v} \times \vec{B})\right]$$

Now if this force displaces a the charge by an elementary amount $d\vec{l}$ then the workdone on the particle is given by

$$\begin{aligned} dW &= F \bullet d\vec{l} = Q\left[\vec{E} + (\vec{v} \times \vec{B})\right] \bullet d\vec{l} \\ &= Q\left[\vec{E} + (\vec{v} \times \vec{B})\right] \bullet \vec{v}dt \\ &= Q\left[\vec{E} \bullet \vec{v}dt\right] + Q\left[(\vec{v} \times \vec{B})\right] \bullet \vec{v}dt \end{aligned}$$

The second part of right hand side ie the work done by the magnetic field on the particle is zero because the force due to the magnetic field is perpendicular to the velocity vector $\vec{v}$. Thus we are only left with

$$\begin{aligned} dW &= Q\left[\vec{E} \bullet \vec{v}dt\right] \\ \frac{dW}{dt} &= Q\,\vec{E} \bullet \vec{v} \end{aligned}$$

The last equation can be further solved with a little bit of dimensional analysis. See charge is coulomb, C and velocity is distance over time ie $\frac{L}{T}$. Hence coulomb per time is current I and current per area is current density $\vec{J}$ and in the numerator the left alone L is getting multiplied by area ie L^2 to give rise to L^3 ie volume V. Thus

$$\frac{dW}{dt} = \vec{E} \bullet \vec{J}V = \int_V \vec{E} \bullet \vec{J}\,dV$$

Now from of the Ampere-Maxwell's Law

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \mu_0 \left[\vec{J} + \epsilon_0 \frac{d\vec{E}}{dt}\right] \\ \frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) &= \vec{J} + \epsilon_0 \frac{d\vec{E}}{dt} \\ \vec{J} &= \frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) - \epsilon_0 \frac{d\vec{E}}{dt} \\ \int_V \vec{E} \bullet \vec{J}\,dV &= \int_V \vec{E} \bullet \left[\frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) - \epsilon_0 \frac{d\vec{E}}{dt}\right] dV \\ &= \frac{1}{\mu_0}\int_V \left[\vec{E} \bullet \left(\vec{\nabla} \times \vec{B}\right)\right] dV - \epsilon_0 \int_V \left(\vec{E} \bullet \frac{d\vec{E}}{dt}\right) dV \\ &= \frac{1}{\mu_0}\int_V \left[\vec{B} \bullet \left(\vec{\nabla} \times \vec{E}\right) - \vec{\nabla} \bullet \left(\vec{E} \times \vec{B}\right)\right] dV - \epsilon_0 \int_V \left(\vec{E} \bullet \frac{d\vec{E}}{dt}\right) dV \qquad \text{(vector idenity, Sem I)} \\ &= \frac{1}{\mu_0}\int_V \left[\vec{B} \bullet \left(-\frac{d\vec{B}}{dt}\right) - \vec{\nabla} \bullet \left(\vec{E} \times \vec{B}\right)\right] dV - \epsilon_0 \int_V \left(\vec{E} \bullet \frac{d\vec{E}}{dt}\right) dV \qquad \text{(Maxwell's 3rd equ}^n\text{)} \end{aligned}$$

$$= \frac{1}{2\mu_0} \int_V \left[-\frac{d\left(B^2\right)}{dt} - \vec{\nabla} \bullet \left(\vec{E} \times \vec{B} \right) \right] dV - \frac{\epsilon_0}{2} \int_V \frac{d\left(E^2\right)}{dt} dV$$

$$= -\frac{d}{dt} \int_V \left[\frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \right] dV - \frac{1}{\mu_0} \int_V \left[\vec{\nabla} \bullet \left(\vec{E} \times \vec{B} \right) \right] dV$$

$$\boxed{\int_V \vec{E} \bullet \vec{J}\, dV = -\frac{d}{dt} \int_V \left[\frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \right] dV - \int_S \frac{1}{\mu_0} \left(\vec{E} \times \vec{B} \right) \bullet dS}$$

which is the work-energy theoram in elctrodynamics. Thus it says that work done by the electric and magnetic fields on the charges within a volume must match the rate of decrease of the energy of the fields within that volume and the net flow of energy into the volume. The big question is what does the net flow of energy into the volume correspond to physically? One possibility is that it might correspond to electromagnetic radiation. The above equation can also be stated as the negative of the work done on the charges within a volume must be equal to the increase in the energy of the electric and magnetic fields within the volume plus the net flow of energy out of the volume.
Usually any difference between the change in energy and the work done is the energy of radiation. This is what is universally presumed in the case of the Poynting theorem, but the empirical evidence is that this cannot be so. If the Poynting vector corresponded to radiation then if a permanent magnet was placed in the vicinity of a body charged with static electricity the combination should glow and is that is not the case.

• **Physical Significances of each term:**

$\int_V \vec{E} \bullet \vec{J}\, dV$: The term $\vec{E} \bullet \vec{J}$ is known as **Joule heating**; it expresses the rate of energy transfer to the charge carriers from the fields. In other words it's the total ohmic power dissipated within the volume. This is the (spatially) local version of an equation with which you are already familiar, P = V I . Notice that this term only contains the electric field because the magnetic field can do no work on the charges.

$\frac{d}{dt} \int_V \left[\frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \right] dV$: The rate at which electromagnetic energy is stored within the volume.

$\int_S \frac{1}{\mu_0} \left(\vec{E} \times \vec{B} \right) \bullet dS$: This term is called **Poynting vector** (it 'Poynts' in the direction of energy transport). The direction of Poynting vector is along the direction of propagation and the magnitude is the rate at which the electromagnetic energy crosses a unit surface area perpendicular to the direction of the vector, ie it is the net flow of energy out of the volume V. But here there is an issue. The issue is what does the net flow of energy out of the volume correspond to physically. You might expect that since the dimensions of the Poynting vector term are energy per unit area per unit time it is the electromagnetic radiation generated in the volume. But there is a major problem with the Poynting vector; it is independent of the charges involved. It is the same whether there is one charge or one hundred million charges, or for that matter, zero charges and at whatever velocities. It can change with time but only as a result of the changes in the electric and magnetic fields. So the Poynting vector term apparently does not correspond to radiation. It is a puzzle as to what it does correspond to but there is no possibility that it corresponds to radiation.

2.7 Relationship between $\vec{E}$ and $\vec{B}$ magnitudes

In an electromagnetic wave, moving along one direction, ie say along z, the magnitudes of electric field and magnetic fields can be expressed as function of plane wave ie

$$\vec{E} = \vec{E_0}\, e^{-i\,(\omega\, t - \kappa\, z)} \qquad \text{and} \qquad \vec{B} = \vec{B_0}\, e^{-i\,(\omega\, t - \kappa\, z)}$$

Now using Maxwell's 3rd relationship $\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$ we get

$$\vec{\nabla} \times \vec{E} = \frac{d}{dz} \left[\vec{E_0}\, e^{-i\,(\omega\, t - \kappa\, z)} \right] = i\,\kappa\,\vec{E}$$

$$\frac{d\vec{B}}{dt} = \frac{d}{dt} \left[\vec{B_0}\, e^{-i\,(\omega\, t - \kappa\, z)} \right] = (-i\,\omega\,\vec{B})$$

Hence we get

$$i\,\kappa\,\vec{E} = -\,(-i\,\omega\,\vec{B})$$

$$\frac{\vec{E_0}\, e^{-i\,(\omega\, t - \kappa\, z)}}{\vec{B_0}\, e^{-i\,(\omega\, t - \kappa\, z)}} = \frac{\omega}{\kappa}$$

$$\boxed{\frac{|\vec{E_0}|}{|\vec{B_0}|} = v \qquad \text{velocity of the wave}}$$

If the wave is travelling in free space then the velocity will be c ie the speed of light.

2.8 Time averaged value of the Poynting Vector, $\frac{1}{\mu_0}(\vec{E}\times\vec{B})$

Unfortunately we cannot blindly apply to power and energy our standard conversion protocol between frequency-domain and time-domain representations because we no longer have only a single frequency present. Time-harmonic power and energy involve the products of sinusoids and therefore exhibit sum and difference frequencies. (Recall superposition of two waves, where you get bandwidth of frequencies as ± terms). That's why we cannot simply represent the Poynting vector $\vec{S}$ for a field at frequency f by Re($\vec{S}\,e^{i\,\omega\,t}$) because power has components at both $f=0$ and 2f, since $\omega = 2\pi f$. Thus what we will do is we can use the convenience of the time-harmonic notation by restricting it to fields, voltages, and currents while representing their products, i.e. powers and energies.
Now assuming the electric and magnetic fields is given by

$$\begin{aligned}\vec{E} &= \vec{E_0}\,e^{i\,\omega\,t} \\ &= (\,E_{Re}\,+i\,E_{Im})\,(cos\,\omega\,t\,+i\,sin\,\omega\,t) \\ &= E_{Re}\,cos\,\omega\,t\,-\,E_{Im}\,sin\,\omega\,t\end{aligned} \qquad \begin{aligned}\vec{B} &= \vec{B_0}\,e^{i\,\omega\,t} \\ &= (\,B_{Re}\,+i\,B_{Im})\,(cos\,\omega\,t\,+i\,sin\,\omega\,t) \\ &= B_{Re}\,cos\,\omega\,t\,-\,B_{Im}\,sin\,\omega\,t\end{aligned}$$

Now the Poynting Vector $\vec{S}$ is given by

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0}\,[(E_{Re}\,cos\,\omega\,t\,-\,E_{Im}\,sin\,\omega\,t)\,\times\,(B_{Re}\,cos\,\omega\,t\,-\,B_{Im}\,sin\,\omega\,t)] \\ &= \frac{1}{\mu_0}\left[(E_{Re}\times B_{Re})\,cos^2\,\omega\,t\,-\,(E_{Re}\times B_{Im})\,cos\,\omega\,t\,sin\,\omega\,t\,-\,(E_{Im}\times B_{Re})\,cos\,\omega\,t\,sin\,\omega\,t\,+\,(E_{Im}\times B_{Im})\,sin^2\,\omega\,t\right]\end{aligned}$$

Now taking the average of the above equation we get

$$\begin{aligned}\mu_0\,<\vec{S}> =& <(E_{Re}\times B_{Re})><cos^2\,\omega\,t> - <(E_{Re}\times B_{Im})><cos\,\omega\,t><sin\,\omega\,t> \\ &- <(E_{Im}\times B_{Re})><cos\,\omega\,t><sin\,\omega\,t> + <(E_{Im}\times B_{Im})><sin^2\,\omega\,t>\end{aligned}$$

As you know that sine and cosine of angles have value lying between $-1$to 1 so the average value of them will be 0 as like 0 lies exactly between [-1,1]. But then taking square of those shifts all the -ve values to the +ve ones so sine squared and cosine squared values will lie then between [0,1]. So, the average value of them will be $\frac{1}{2}$. Hence we will get

$$\begin{aligned}<\vec{S}> &= \frac{1}{\mu_0}\left[\frac{1}{2}<(E_{Re}\times B_{Re})>-0-\,0+\frac{1}{2}<(E_{Im}\times B_{Im})>\right] \\ &= \frac{1}{2\mu_0}\,[<(E_{Re}\times B_{Re})>+<(E_{Im}\times B_{Im})>]\end{aligned}$$

But to compute the Poynting vector the simplest way to use a real form for the both fields $\vec{E}$ and $\vec{B}$ rather than a complex exponential representation.

2.8.1 Energy contribution from the $\vec{E}$ and $\vec{B}$ fields

Electromagnetic waves bring energy into a system by virtue of their electric and magnetic fields. These fields can exert forces and move charges in the system and, thus, do work on them. Clearly, the larger the strength of the electric and magnetic fields, the more work they can do and the greater the energy the electromagnetic wave carries. The wave energy is determined by the wave amplitude. But by how much amount does each field contribute to the wave energy? Let us look at that.
We have Poynting vector which speaks about the flux of energy through any surface in a direction perpendicular to both $\vec{E}$ and $\vec{B}$ fields as

$$\begin{aligned}\int_S \vec{S}\bullet d\vec{S} &= \frac{1}{\mu_0}\int_V\left[\vec{\nabla}\bullet\left(\vec{E}\times\vec{B}\right)\right]\,dV \\ &= \frac{1}{\mu_0}\int_V\left[\vec{B}\bullet(\vec{\nabla}\times\vec{E})\,-\,\vec{E}\bullet(\vec{\nabla}\times\vec{B})\right]\,dV \\ &= \frac{1}{\mu_0}\int_V\left[\vec{B}\bullet\left(-\frac{d\vec{B}}{dt}\right)\,-\,\vec{E}\bullet\left(\mu_0\,\epsilon_0\,\frac{d\vec{E}}{dt}\right)\right]\,dV\end{aligned}$$

This has been found by using Maxwell's 3rd and 4th relationships along with putting $\sigma=0$ (for free space). On further simplifying we get

$$\int_S \vec{S}\bullet d\vec{S} = -\frac{d}{dt}\int_V\left[\frac{\epsilon_0\,E^2}{2}\,+\,\frac{B^2}{2\mu_0}\right]\,dV$$

Thus the energy in any part of the electromagnetic wave is the sum of the energies of the electric and magnetic fields. Or equivalently saying the energy per unit volume, or energy density u, is the sum of the energy density from the electric field and the energy density from the magnetic field. Now the ratio of energy contribution from these two fields are

$$\begin{aligned} \frac{U_E}{U_M} &= \frac{\frac{\epsilon_0 E^2}{2}}{\frac{B^2}{2\mu_0}} = \frac{\mu_0 \epsilon_0 E^2}{B^2} = \mu_0 \epsilon_0 c^2 = 1 \\ U_E &= U_M \end{aligned}$$

This shows that the magnetic energy density U_M and electric energy density U_E are equal, despite the fact that changing electric fields generally produce only small magnetic fields.

Chapter 3

Reflection, Refraction (Transmission) and Polarization of Electro-Magnetic Waves

3.1 Introduction

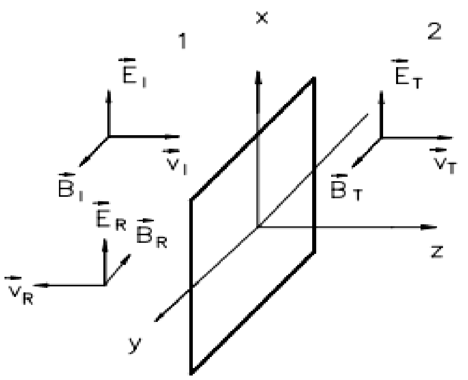
We have so far discussed the propagation of electromagnetic wave in an isotropic, homogeneous, dielectric medium, such as in air or vacuum. In this chapter, we will discuss what happens when a plane electromagnetic wave is incident at the interface between two dielectric media. For being specific, you can will take one of the medium to be air or vacuum and the other to be a dielectric such as glass. We have come across in such a situation is the phenomenon of reflection, refraction and transmission of light waves at such an interface. But here, we will investigate this problem from the point of view of electromagnetic theory.

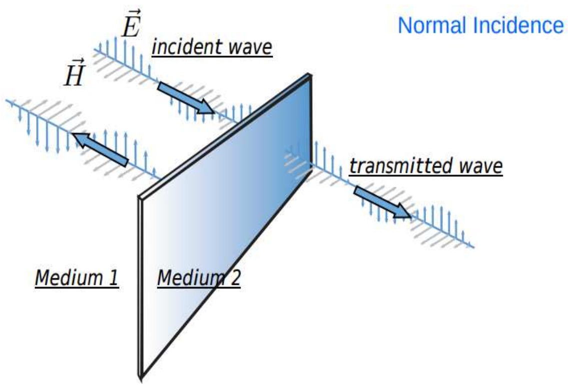
• **Key points to be remembered**:

Point 1: We will always assume a plane wave propagating in medium 1, with permittivity ϵ_1 and permeability μ_1 encounters an interface with a different medium 2, with permittivity ϵ_2 and permeability μ_2, a portion of the wave is reflected back to the medium 1 from the interface while the remainder of the wave is transmitted to the medium 2.
Point 2: The wavenumbers of incident electric/magnetic field's plane wave solution $\vec{E_I}$ or $\vec{B_I}$ and their reflected plane wave solution ie $\vec{E_R}$ or $\vec{B_R}$ are the same because both waves are in the Medium 1.
Point 3: The wavenumber of transmitted electric/magnetic field's plane wave solution $\vec{E_T}$ or $\vec{B_T}$ is different since it is in a dierent medium ie in medium 2.
Point 4: The angular frequencies of all the waves are of course the same as frequency does not depend on the medium.
Point 5: You should also be caution with the sign of the ($\omega t-$ term) which indicates the propagation direction of the respective wave.

3.2 Reflection, and Transmission of EM Waves at a boundary (interface) of two media in normal incidence

Assume an incident light with $\vec{E}$ polarized in the x-direction and $\vec{\kappa}$ (or $\vec{v}$) in z direction entering from medium 1 to medium 2. The normal of the boundary surface is in the z-direction. Let us choose the interface to be the xy plane ($z = 0$).





For the incident wave | For the reflected wave | For the transmitted wave

$$\text{The E field} \quad \vec{E_I} = \vec{E_{01}}\, e^{-i(\omega t - \kappa_1 z)}\hat{x} \qquad \vec{E_R} = \vec{E_{01}}\, e^{-i(\omega t + \kappa_1 z)}\hat{x} \qquad \vec{E_T} = \vec{E_{02}}\, e^{-i(\omega t - \kappa_2 z)}\hat{x}$$

$$\text{The B field} \quad \vec{B_I} = \vec{B_{01}}\, e^{-i(\omega t - \kappa_1 z)}\hat{y} \qquad \vec{B_R} = \vec{B_{01}}\, e^{-i(\omega t + \kappa_1 z)}\hat{y} \qquad \vec{B_T} = \vec{B_{02}}\, e^{-i(\omega t - \kappa_2 z)}\hat{y}$$

$$= \frac{\vec{E_{01}}}{v_1}\, e^{-i(\omega t - \kappa_1 z)}\hat{y} \qquad = \frac{-\vec{E_{01}}}{v_1}\, e^{-i(\omega t + \kappa_1 z)}\hat{y} \qquad = \frac{\vec{E_{02}}}{v_2}\, e^{-i(\omega t - \kappa_1 z)}\hat{y}$$

Our job now is to use boundary conditions to find the complex amplitudes of the reflected and transmitted waves in terms of that of incident wave. So, using the boundary condition that the tangential component of electric and magnetic field is continous ie $\vec{E_{||}}_1 = \vec{E_{||}}_2$ and $\vec{B_{||}}_1 = \vec{B_{||}}_2$ at the interface of the two media ie at z= 0.

For the electric field at z= 0, ie on the boundary the time varying terms, $e^{-i\omega t}$, are the same for all fields. this will immediately give us

$$E_{0I} + E_{0R} = E_{0T} \qquad \text{(the orientation of the E field stays the same)} \qquad .....(I)$$

Now for the magnetic field at z= 0, ie on the boundary

$$\frac{1}{\mu_1} B_{0I} - \frac{1}{\mu_1} B_{0R} = \frac{1}{\mu_2} B_{0T} \qquad \text{(the orientation of the B field reverses)}$$

$$\frac{1}{\mu_1 v_1} E_{0I} - \frac{1}{\mu_1 v_1} E_{0R} = \frac{1}{\mu_2 v_2} E_{0T}$$

$$E_{0I} - E_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2} E_{0T} = \gamma\, E_{0T} \qquad \left[\gamma = \frac{\mu_1 v_1}{\mu_2 v_2}\right] \qquad .....(II)$$

Now adding (I) & (II) we get

$$2\, E_{0I} = (1 + \gamma)\, E_{0T} \qquad .....(C)$$

And substracting (II) from (I) we get

$$2\, E_{0R} = (1 - \gamma)\, E_{0T} \qquad .....(D)$$

Now dividing (D) by (C) we get

$$\frac{2\, E_{0R}}{2\, E_{0I}} = \frac{1 - \gamma}{1 + \gamma}$$

$$\frac{E_{0R}}{E_{0I}} = \frac{\left[1 - \frac{\mu_1 v_1}{\mu_2 v_2}\right]}{\left[1 + \frac{\mu_1 v_1}{\mu_2 v_2}\right]} = \frac{\mu_2 v_2 - \mu_1 v_1}{\mu_2 v_2 + \mu_1 v_1} = \frac{\sqrt{\frac{\mu_2}{\epsilon_2}} - \sqrt{\frac{\mu_1}{\epsilon_1}}}{\sqrt{\frac{\mu_2}{\epsilon_2}} + \sqrt{\frac{\mu_1}{\epsilon_1}}}$$

For free space $\mu_1 = \mu_2 = \mu_0$ (something similar to assume as non-magnetic media, $\mu_{r_1} = \mu_{r_2} = 1$) then solving out the above equation will lead to

$$\frac{E_{0R}}{E_{0I}} = \frac{\sqrt{\epsilon_1} - \sqrt{\epsilon_2}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}}$$

$$= \frac{n_1 - n_2}{n_1 + n_2} \qquad \text{(where n s are the refractive indices)}$$

The **coefficient of reflection**, R, is defined as the ratio of the intensities (nothing but the amplitude squared) of the reflected and incident waves

$$\boxed{R = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$$

Now replacing E_{0R} from (D) and putting it in (I) we get

$$E_{0I} + \left(\frac{1 - \gamma}{2}\right) E_{0T} = E_{0T}$$

$$E_{0I} = \frac{1 + \gamma}{2} E_{0T}$$

$$\frac{E_{0T}}{E_{0I}} = \frac{2}{1 + \gamma} = \frac{2}{1 + \frac{\mu_1 v_1}{\mu_2 v_2}} = \frac{2\, \mu_2 v_2}{\mu_2 v_2 + \mu_1 v_1} = \frac{2\sqrt{\frac{\mu_2}{\epsilon_2}}}{\sqrt{\frac{\mu_2}{\epsilon_2}} + \sqrt{\frac{\mu_1}{\epsilon_1}}} = \frac{2\, n_2}{n_1 + n_2} \qquad \text{(n = refractive indices)}$$

The **coefficient of transmission**, T, is defined as the ratio of the intensities (nothing but the amplitude squared) of the transmitted and incident waves

$$\boxed{T = \left(\frac{E_{0T}}{E_{0I}}\right)^2 = \left(\frac{2\, n_2}{n_1 + n_2}\right)^2}$$

3.2.1 Value of the Reflection (R) and Transmission (T) coefficient in terms of Poynting vector

Let us assume S_I, S_R and S_T be the Ponynting vector associated with incident, reflected and transmitted wave respectively. Now

For the incident wave

$$S_I = \frac{1}{\mu_1}\left(\vec{E_{0I}} \times \vec{B_{0I}}\right) = \frac{E_{0I}^2}{\mu_1 v_1}$$

For the reflected wave

$$S_R = \frac{1}{\mu_1}\left(\vec{E_{0R}} \times \vec{B_{0R}}\right) = \frac{E_{0R}^2}{\mu_1 v_1}$$

For the transmitted wave

$$S_T = \frac{1}{\mu_2}\left(\vec{E_{0T}} \times \vec{B_{0T}}\right) = \frac{E_{0T}^2}{\mu_2 v_2}$$

Now the coefficients are calculated as follows

The reflection coefficient

$$R = \frac{S_R}{S_I} = \frac{\frac{E_{0R}^2}{\mu_1 v_1}}{\frac{E_{0I}^2}{\mu_1 v_1}}$$

$$R = \frac{E_{0R}^2}{E_{0I}^2} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

The transmission coefficient

$$T = \frac{S_T}{S_I} = \frac{\frac{E_{0T}^2}{\mu_2 v_2}}{\frac{E_{0I}^2}{\mu_1 v_1}}$$

$$T = \frac{\mu_1 v_1}{\mu_2 v_2}\frac{E_{0T}^2}{E_{0I}^2} = \frac{n_2}{n_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

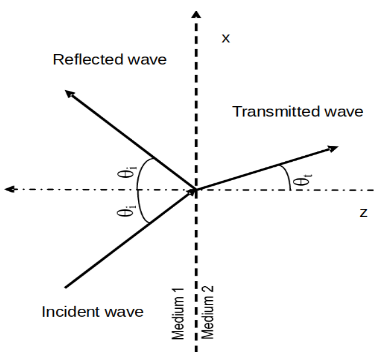
Key Points to be taken away:
- $\vec{E_T}$ and $\vec{E_I}$ are always in phase.
- If $n_1 > n_2$ (glass to air), $\vec{E_R}$ and $\vec{E_I}$ are in phase.
- If $n_1 < n_2$ (air to glass), $\vec{E_R}$ and $\vec{E_I}$ are out of phase by 180^o.
- It is easy to show that $R + T = 1$, satisfying the energy conservation law. This is true even if we do not assume $\mu_1 \sim \mu_2 \sim \mu_0$.

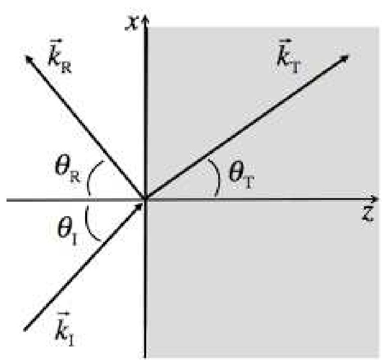
If light is going from air ($n_1 = 1$) to glass ($n_1 = 1.5$), the transmitted amplitude will be 80% of the incident amplitude, and the reflected amplitude will be 20% of the incident amplitude. The transmitted flux density will be 96% of the incident flux density, and the reflected flux density will be 4 percent of the incident flux density. If $n_1 = n_2$ there will be no reflection at the boundary; in effect there is no boundary. The larva of the midge ***Chaoborus****, known as the Phantom Midge, is an aquatic creature whose body has a refractive index equal to the refractive index of water. The picture shows a photograph of one of them in the water. (If you dont believe me, look it up on the Web.)*

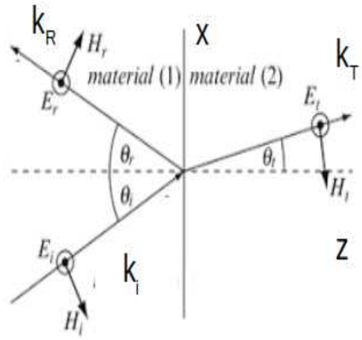

Larva of the Phantom Midge, *Chaoborus* sp.

3.3 Reflection and Transmission of EM Waves for oblique incidence : Laws of Reflection and Refraction

Let us consider a plane wave that obliquely incidents at the boundary of two media that are characterized by their permittivity (ϵ) and permeability (μ). Select the z-axis normal to the boundary and the incident wave vector $\vec{k_I}$ on the xz plane. We do not assume any particular directions of wave vectors for the reflected $\vec{k_R}$ and transmitted $\vec{k_T}$. The wave frequencies for all waves are the same and are determined by the source.







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For the incident wave | For the reflected wave | For the transmitted wave

The E field $\vec{E_I} = \vec{E_{01}}\, e^{-i(\omega t - \kappa_I r)}$ $\quad \vec{E_R} = \vec{E_{01}}\, e^{-i(\omega t - \kappa_R r)}$ $\quad \vec{E_T} = \vec{E_{0R}}\, e^{-i(\omega t - \kappa_T r)}$

The B field $\vec{B_I} = \frac{1}{\omega_1}(\kappa_I \times \vec{E_I})$ $\quad \vec{B_R} = \frac{1}{\omega_1}(\kappa_R \times \vec{E_R})$ $\quad \vec{B_T} = \frac{1}{\omega_2}(\kappa_T \times \vec{E_T})$

One thing here is clear that if you look at the extreme right figure after reflection or refraction there has been change in the orientation of the field variables, which points towards the change of phase of in the event. Thus in order to deal with this problem we have to deal with the phase part of the equations.

• **The first law of reflection (or refraction)**:

The incident ray, the reflected ray and the transmitted ray (ie the refracted ray) remain in the same plane.

Proof

At the boundary of the interface ie z=0 all the field variables must coincide. Mathematically it equivalent to write in the following way

$$
\begin{aligned}
\vec{E_{01}}\, e^{-i(\omega t - \kappa_I r)} &= \vec{E_{01}}\, e^{-i(\omega t - \kappa_R r)} = \vec{E_{02}}\, e^{-i(\omega t - \kappa_T r)} \qquad \text{(at z=0)} \\
e^{-i(\omega t - \kappa_I r)} &= e^{-i(\omega t - \kappa_R r)} = e^{-i(\omega t - \kappa_T r)} \qquad \text{(since dealing only with phase)} \\
e^{i\kappa_I r} &= e^{i\kappa_R r} = e^{i\kappa_T r} \\
\vec{\kappa_I}\bullet\vec{r} &= \vec{\kappa_R}\bullet\vec{r} = \vec{\kappa_T}\bullet\vec{r}
\end{aligned}
$$

$$(\kappa_{I_x}\hat{i} + \kappa_{I_y}\hat{j} + \kappa_{I_z}\hat{k})\bullet(x\hat{i} + y\hat{j} + z\hat{k}) = (\kappa_{R_x}\hat{i} + \kappa_{R_y}\hat{j} + \kappa_{R_z}\hat{k})\bullet(x\hat{i} + y\hat{j} + z\hat{k}) = (\kappa_{T_x}\hat{i} + \kappa_{T_y}\hat{j} + \kappa_{T_z}\hat{k})\bullet(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\kappa_{I_x} x + \kappa_{I_y} y = \kappa_{R_x} x + \kappa_{R_y} y = \kappa_{T_x} x + \kappa_{T_y} y$$

The last equation is a linnear indentity which can only be true iff

$$\kappa_{I_x} = \kappa_{R_x} = \kappa_{T_x} \qquad \text{and} \qquad \kappa_{I_y} = \kappa_{R_y} = \kappa_{T_y}$$

Hence this expression indicates that the incident, reflected and refracted wave remain in the same plane ie in this case xy-plane. Had the propagation been in y direction it would have been xz-plane and likewise. In this way it can be proved.

• **The second law of reflection**:

The angle of incidence is equal to the angle of reflection

Proof

Now decomposing the propagation vector $\vec{\kappa}$ into its component we get

For the incident wave | For the reflected wave | For the transmitted wave

$\vec{\kappa_I} = \kappa_I\, sin\theta_I \hat{i} + \kappa_I\, cos\theta_I \hat{k}$ $\quad \vec{\kappa_R} = \kappa_R\, sin\theta_R \hat{i} - \kappa_R\, cos\theta_R \hat{k}$ $\quad \vec{\kappa_T} = \kappa_T\, sin\theta_T \hat{i} + \kappa_T\, cos\theta_T \hat{k}$

Now at the boundary ie at z=0

$$
\begin{aligned}
\vec{\kappa_I}\bullet\vec{r} &= \vec{\kappa_R}\bullet\vec{r} = \vec{\kappa_T}\bullet\vec{r} \\
(\kappa_I\, sin\theta_I \hat{i} + \kappa_I\, cos\theta_I \hat{k})\bullet(x\hat{i} + y\hat{j} + z\hat{k}) &= (\kappa_R\, sin\theta_R \hat{i} - \kappa_R\, cos\theta_R \hat{k})\bullet(x\hat{i} + y\hat{j} + z\hat{k}) \\
\kappa_I\, sin\theta_I\, x &= \kappa_R\, sin\theta_R\, x \qquad \text{(at z=0 and at same medium } \kappa_I = \kappa_R) \\
sin\theta_I &= sin\theta_R \\
\theta_I &= \theta_R
\end{aligned}
$$

Thus the angle of incidence is found out to be equal to angle of reflection.

• **The second law of refraction**:

The ratio of sine of angle of incidence to the sine of angle of refraction is always a constant quantity. This constant is called as refractive index of the medium 2 w.r.to the medium 1. This law is also known as Snell's law.

Proof

Now decomposing the propagation vector $\vec{\kappa}$ into its component we get

For the incident wave | For the reflected wave | For the transmitted wave

$\vec{\kappa_I} = \kappa_I\, sin\theta_I \hat{i} + \kappa_I\, cos\theta_I \hat{k}$ $\quad \vec{\kappa_R} = \kappa_R\, sin\theta_R \hat{i} - \kappa_R\, cos\theta_R \hat{k}$ $\quad \vec{\kappa_T} = \kappa_T\, sin\theta_T \hat{i} + \kappa_T\, cos\theta_T \hat{k}$

Now at the boundary ie at z=0

$$\vec{\kappa_I}\bullet\vec{r} = \vec{\kappa_R}\bullet\vec{r} = \vec{\kappa_T}\bullet\vec{r}$$

$$(\kappa_I\, sin\theta_I\hat{i} + \kappa_I\, cos\theta_I\hat{k})\bullet(x\hat{i} + y\hat{j} + z\hat{k}) = (\kappa_T\, sin\theta_T\hat{i} + \kappa_T\, cos\theta_T\hat{k})\bullet(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\kappa_I\, sin\theta_I\, x = \kappa_T\, sin\theta_T\, x \qquad \text{(at z=0)}$$

$$\frac{sin\theta_I}{sin\theta_T} = \frac{\kappa_T}{\kappa_I} = \frac{\omega\sqrt{\mu_2\,\epsilon_2}}{\omega\sqrt{\mu_1\,\epsilon_1}} = \frac{n_2}{n_1} \qquad \text{(For non magnetic material } \mu = 1\text{)}$$

Thus the Snell's law can be proved.

3.4 Fresnel Equations

The Fresnel equations relate the amplitudes, phases, and polarizations of the transmitted and reflected waves of electric fields to the corresponding parameters of the incident waves of electric field (the waves' magnetic fields can also be related using similar coefficients) which emerges when light enters an interface between two media with different indices of refraction. When light strikes the interface between a medium with refractive index n_1 and a second medium with refractive index n_2, both reflection and refraction of the light may occur. In fact, the intensity of light reflected from the surface of a dielectric, as a function of the angle of incidence was first obtained by Fresnel in 1823, as a part of his comprehensive wave theory of light. However, the Fresnel equations are fully consistent with the rigorous treatment of light in the framework of Maxwell equations. But while deriving the equations few assumptions were made. These are as follows

Assumptions:

- The interface between the media is flat, homogeneous and isotropic.
- The incident light is assumed to be a plane wave, since any incident light field can be decomposed into plane waves and be made polarized.
- Both the media are non-magnetic so that the permeability of both media are the same.
- The two media differ by their dielectric constant, the incident medium may also be taken as air.
- We further assume that there are no free charges or currents on the surface of interface between the two media.

Electromagnetic waves follow the superposition principle. In order to simplify the math associate with our problem and derive the Fresnel equation, we split the incoming EM wave into two modes

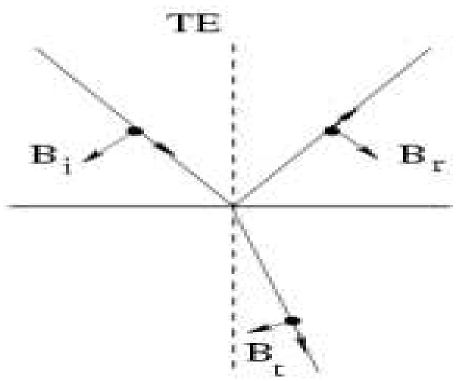
Mode 1:
the first case where the electric fields are perpendicular to the plane of incidence. This is case is also called as **Transverse Electric** and is represented by *s - polarization*, s standing for a German word **"senkrecht"** meaning perpendicular.

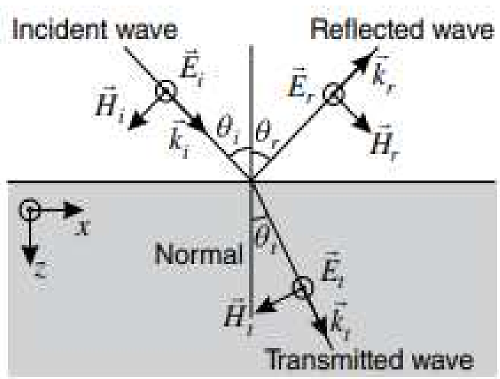
Mode 2:
This case is known as *p - polarization*, p standing for **"parallel"**, where the electric field is parallel to the incident plane. A case known as **Transverse Magnetic** (since parallel E-field will guarantee perpendicular B-field to the plane of incidence).

Let us now derive the Reflection (R) and Transmission (T) coefficient for both the two cases.

3.4.1 When the $\vec{E}$ is perpendicular to the plane of incidence: Transverse Electric

To derive the Fresnel equations, consider two optical media separated by an interface, as shown in Fig. A plane optical wave is propagating toward the interface with propagation vector $\vec{\kappa_I}$ oriented at angle θ_I with respect to the interface normal. The electric field amplitude of the wave is given by $\vec{E_I}$. On incidence onto the interface, this wave will be partially transmitted and partially reflected. The transmitted wave will propagate at angle θ_T and the reflected angle will be θ_R. We denote the amplitudes of these two waves as $\vec{E_T}$ and $\vec{E_R}$, respectively. Our goal is to determine these amplitudes.





To accomplish this, we apply the boundary conditions for the electric and magnetic fields at an interface between two media with different electromagnetic properties.

1: The parallel component of $\vec{E}$ is continuous across the boundary between the two media.
2: The perpendicular component of $\vec{B}$ (but parallel component of $\frac{\vec{B}}{\mu}$) is continuous across the boundary between the two media.

• The value of reflection coefficient, R

The boundary condition 1 will give

$$E_I + E_R = E_T \qquad \text{.............(eqn. 1)}$$

While for the magnetic field the second one will give

$$\begin{aligned}
\frac{1}{\mu_1} B_I\, cos\theta_I - \frac{1}{\mu_1} B_R\, cos\theta_R &= \frac{1}{\mu_2} B_T\, cos\theta_T \\
\frac{\kappa_I \times E_I}{\mu_1\, \omega_1} cos\theta_I - \frac{\kappa_R \times E_R}{\mu_1\, \omega_1} cos\theta_R &= \frac{\kappa_T \times E_T}{\mu_2\, \omega_2} cos\theta_T \\
\frac{\sqrt{\mu_1\, \epsilon_1}\, E_I}{\mu_1} cos\theta_I - \frac{\sqrt{\mu_1\, \epsilon_1}\, E_R}{\mu_1} cos\theta_R &= \frac{\sqrt{\mu_2\, \epsilon_2}\, E_T}{\mu_2} cos\theta_T \\
\sqrt{\frac{\epsilon_1}{\mu_1}}\, E_I\, cos\theta_I - \sqrt{\frac{\epsilon_1}{\mu_1}}\, E_R\, cos\theta_I &= \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_T\, cos\theta_T \qquad \text{(since } \theta_I = \theta_R) \\
\sqrt{\frac{\epsilon_1}{\mu_1}}\, (E_I - E_R)\, cos\theta_I &= \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_T\, cos\theta_T \\
\sqrt{\frac{\epsilon_1}{\mu_1}}\, (E_I - E_R)\, cos\theta_I &= \sqrt{\frac{\epsilon_2}{\mu_2}}\, (E_I + E_R)\, cos\theta_T \qquad \text{(by virtue of eqn. 1)} \\
\frac{(E_I - E_R)}{(E_I + E_R)} &= \frac{\sqrt{\frac{\epsilon_2}{\mu_2}}\, cos\theta_T}{\sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I} \\
\frac{(E_I - E_R) + (E_I + E_R)}{(E_I - E_R) - (E_I + E_R)} &= \frac{\sqrt{\frac{\epsilon_2}{\mu_2}}\, cos\theta_T + \sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I}{\sqrt{\frac{\epsilon_2}{\mu_2}}\, cos\theta_T - \sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I} \\
\frac{-2\, E_I}{2\, E_R} &= \frac{\sqrt{\frac{\epsilon_2}{\mu_2}}\, cos\theta_T + \sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I}{\sqrt{\frac{\epsilon_2}{\mu_2}}\, cos\theta_T - \sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I} \\
\frac{E_I}{E_R} &= \frac{\sqrt{\frac{\epsilon_2}{\mu_2}}\, cos\theta_T + \sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I}{\sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I - \sqrt{\frac{\epsilon_2}{\mu_2}}\, cos\theta_T}
\end{aligned}$$

$$\boxed{R = \frac{E_R}{E_I} = \frac{\sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I - \sqrt{\frac{\epsilon_2}{\mu_2}}\, cos\theta_T}{\sqrt{\frac{\epsilon_2}{\mu_2}}\, cos\theta_T + \sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I}} \qquad \text{Fresnel equation 1}$$

Now if we assume $\mu_1 = \mu_2 = \mu_0$ ie the permeability for free space and taking ϵ_1 common the above equation will take shape of

$$R = \frac{cos\theta_I - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\, cos\theta_T}{cos\theta_I + \sqrt{\frac{\epsilon_2}{\epsilon_1}}\, cos\theta_T} = \frac{cos\theta_I - \frac{n_2}{n_1}\, cos\theta_T}{cos\theta_I + \frac{n_2}{n_1}\, cos\theta_T} \qquad \text{(refractive index, } n = \sqrt{\epsilon_r})$$

$$\boxed{R = \frac{n_1\, cos\theta_I - n_2\, cos\theta_T}{n_1\, cos\theta_I + n_2\, cos\theta_T}} \qquad \text{(Another form of Fresnel equation 1a)} \qquad \text{.............(A)}$$

- **The value of transmission coefficient, T**

Once this equation is derived you can start from anywhere from the continuity point of view. But I will start right from the scratch since in your exam question may come asking to derive any one of them.

The boundary condition 1 will give

$$E_I + E_R = E_T \qquad \text{.............(eqn..1(a))}$$

While for the magnetic field the second one will give

$$\frac{1}{\mu_1} B_I\, cos\theta_I - \frac{1}{\mu_1} B_R\, cos\theta_R = \frac{1}{\mu_2} B_T\, cos\theta_T$$

$$\frac{\kappa_I \times E_I}{\mu_1\,\omega_1} cos\theta_I - \frac{\kappa_R \times E_R}{\mu_1\,\omega_1} cos\theta_R = \frac{\kappa_T \times E_T}{\mu_2\,\omega_2} cos\theta_T$$

$$\frac{\sqrt{\mu_1\,\epsilon_1}\,E_I}{\mu_1} cos\theta_I - \frac{\sqrt{\mu_1\,\epsilon_1}\,E_R}{\mu_1} cos\theta_R = \frac{\sqrt{\mu_2\,\epsilon_2}\,E_T}{\mu_2} cos\theta_T$$

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\, E_I\, cos\theta_I - \sqrt{\frac{\epsilon_1}{\mu_1}}\, E_R\, cos\theta_I = \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_T\, cos\theta_T \qquad \text{(since } \theta_I = \theta_R\text{)}$$

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\,(E_I - E_R)\, cos\theta_I = \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_T\, cos\theta_T$$

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\,[E_I - (E_T - E_I)]\, cos\theta_I = \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_T\, cos\theta_T \qquad \text{(by virtue of eqn..(1a))}$$

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\,(2\,E_I - E_T)\, cos\theta_I = \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_T\, cos\theta_T$$

$$2\sqrt{\frac{\epsilon_1}{\mu_1}}\, E_I\, cos\theta_I - \sqrt{\frac{\epsilon_1}{\mu_1}}\, E_T\, cos\theta_I = \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_T\, cos\theta_T$$

$$2\sqrt{\frac{\epsilon_1}{\mu_1}}\, E_I\, cos\theta_I = \sqrt{\frac{\epsilon_1}{\mu_1}}\, E_T\, cos\theta_I + \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_T\, cos\theta_T$$

$$\boxed{T = \frac{E_T}{E_I} = \frac{2\sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I}{\sqrt{\frac{\epsilon_1}{\mu_1}}\, cos\theta_I + \sqrt{\frac{\epsilon_2}{\mu_2}}\, cos\theta_T}} \qquad \text{Fresnel equation 1b}$$

Now if we assume $\mu_1 = \mu_2 = \mu_0$ ie the permeability for free space and taking ϵ_1 common the above equation will take shape of

$$T = \frac{2\,cos\theta_I}{cos\theta_I + \sqrt{\frac{\epsilon_2}{\epsilon_1}}\, cos\theta_T}$$

$$= \frac{2\,cos\theta_I}{cos\theta_I + \sqrt{\frac{n_2}{n_1}}\, cos\theta_T} \qquad \text{(refractive index, } n = \sqrt{\epsilon_r}\text{)}$$

$$\boxed{T = \frac{2\,n_1\,cos\theta_I}{n_1\,cos\theta_I + n_2\,cos\theta_T}} \qquad \text{(Another form of Fresnel equation 1b)} \qquad \text{.............(B)}$$

Key points to be taken away:

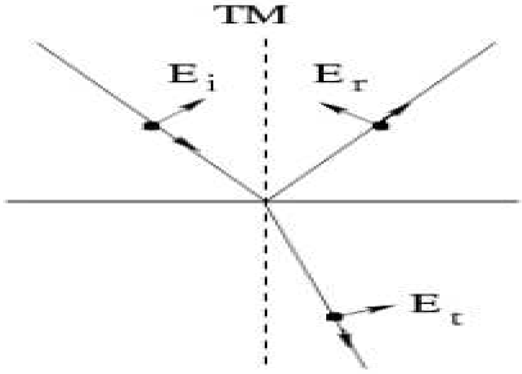
1. Both the coefficients (R & T) are independant of the material properties ie permittivity and permeability (as per the second form of the equations), though still have the implication of the refractive indices.

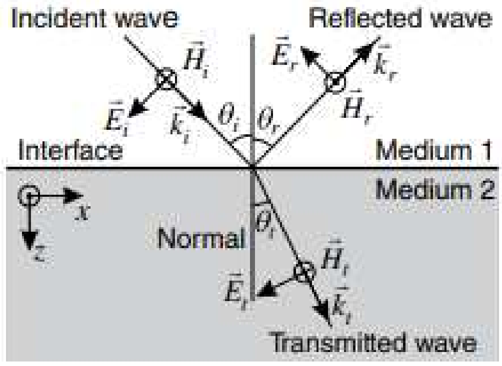
2. Both the coefficients (R & T) are only dependant on the angle of incidence θ_I and angle of refraction (transmission)θ_R (as per the both form of the equations).

3.4.2 When the $\vec{E}$ is parallel to the plane of incidence: Transverse Magnetic

In order to derive this set of equation we will just use the principle of reversability of light. In such cases the angles will straightway change from incident to refracted and vice versa. Also the boundary conditions for the electric field will now be the boundary conditions for the magnetic and likewise for magnetic to electric.

To derive the Fresnel equations, consider two optical media separated by an interface, as shown in Fig. A plane optical wave is propagating toward the interface with propagation vector $\vec{\kappa_I}$ oriented at angle θ_I with respect to the interface normal. The electric field amplitude of the wave is given by $\vec{B_I}$. On incidence onto the interface, this wave will be partially transmitted and partially reflected. The transmitted wave will propagate at angle θ_T and the reflected angle will be θ_R. We denote the amplitudes of these two waves as $\vec{B_T}$ and $\vec{B_R}$, respectively. Our goal is to determine these amplitudes.





To accomplish this, we apply the boundary conditions for the electric and magnetic fields at an interface between two media with different electromagnetic properties.

1: The parallel component of $\frac{\vec{B}}{\mu}$ is continuous across the boundary between the two media.

2: The perpendicular component of $\vec{E}$ is continuous across the boundary between the two media.

• **The value of reflection coefficient, R**

The boundary condition 1 will give

$$\begin{aligned}
\frac{B_I}{\mu_1} + \frac{B_R}{\mu_1} &= \frac{B_T}{\mu_2} \\
\frac{E_I}{\mu_1\, c} + \frac{E_R}{\mu_1\, c} &= \frac{E_T}{\mu_2\, c} \\
\sqrt{\frac{\epsilon_1}{\mu_1}}(E_I + E_R) &= \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_T && \text{since } c = \frac{1}{\sqrt{\epsilon\,\mu}} \\
n_1\,(E_I + E_R) &= n_2\, E_T && \text{assuming } \mu_1 = \mu_2 = \mu_0 \text{ and refractive index} = \sqrt{\epsilon} \\
E_I + E_R &= \frac{n_2}{n_1}\, E_T && \text{.............(eqn..1)}
\end{aligned}$$

While for the electric field the second one will give

$$\begin{aligned}
E_I\, cos\theta_I - E_R\, cos\theta_R &= E_T\, cos\theta_T \\
(E_I - E_R)\, cos\theta_I &= E_T\, cos\theta_T && \text{since } \theta_I = \theta_R \\
E_I - E_R &= \frac{cos\theta_T}{cos\theta_I}\, E_T && \text{.............(eqn..2)}
\end{aligned}$$

Now dividing the eqn.(1) by eqn.(2) we get

$$\begin{aligned}
\frac{E_I + E_R}{E_I - E_R} &= \frac{\frac{n_2}{n_1}\, E_T}{\frac{cos\theta_T}{cos\theta_I}\, E_T} = \frac{n_2\, cos\theta_I}{n_1\, cos\theta_T} \\
\frac{(E_I + E_R) + (E_I - E_R)}{(E_I + E_R) - (E_I - E_R)} &= \frac{n_2\, cos\theta_I + n_1\, cos\theta_T}{n_2\, cos\theta_I - n_1\, cos\theta_T} \\
\frac{E_I}{E_R} &= \frac{n_2\, cos\theta_I + n_1\, cos\theta_T}{n_2\, cos\theta_I - n_1\, cos\theta_T}
\end{aligned}$$

$$\boxed{R = \frac{E_R}{E_T} = \frac{n_2\, cos\theta_I - n_1\, cos\theta_T}{n_2\, cos\theta_I + n_1\, cos\theta_T}} \qquad \text{(Fresnel equation)} \qquad \text{.............(C)}$$

- **The value of transmission coefficient, T**

Once this equation is derived you can start from anywhere from the continuity point of view. But I will start right from the scratch since in your exam question may come asking to derive any one of them.

The boundary condition 1 $\frac{B_I}{\mu_1} + \frac{B_R}{\mu_1} = \frac{B_T}{\mu_2}$ has lead us to the following simplification (see the last section)

$$\begin{aligned}\frac{B_I}{\mu_1} + \frac{B_R}{\mu_1} &= \frac{B_T}{\mu_2}\\ E_I + E_R &= \frac{n_2}{n_1} E_T\\ E_R &= \frac{n_2}{n_1} E_T - E_I \qquad \text{.............(eqn..1)}\end{aligned}$$

Now replacing the value of this E_R in the eqn..(2) of the last section we will find

$$\begin{aligned}E_I - E_R &= \frac{cos\theta_T}{cos\theta_I} E_T\\ E_I - (\frac{n_2}{n_1} E_T - E_I) &= \frac{cos\theta_T}{cos\theta_I} E_T\\ 2\,E_I &= \left(\frac{cos\theta_T}{cos\theta_I} + \frac{n_2}{n_1}\right) E_T\\ &= \left(\frac{n_1\,cos\theta_T + n_2\,cos\theta_I}{n_1\,cos\theta_I}\right) E_T\\ \boxed{T = \frac{E_T}{E_I} = \frac{2\,n_1\,cos\theta_I}{n_1\,cos\theta_T + n_2\,cos\theta_I}} &\qquad \text{(Fresnel equation)} \qquad \text{.............(D)}\end{aligned}$$

Key points to be taken away:

1. Both the coefficients (R & T) are independant of the material properties ie permittivity and permeability (as per the second form of the equations), though still have the implication of the refractive indices.

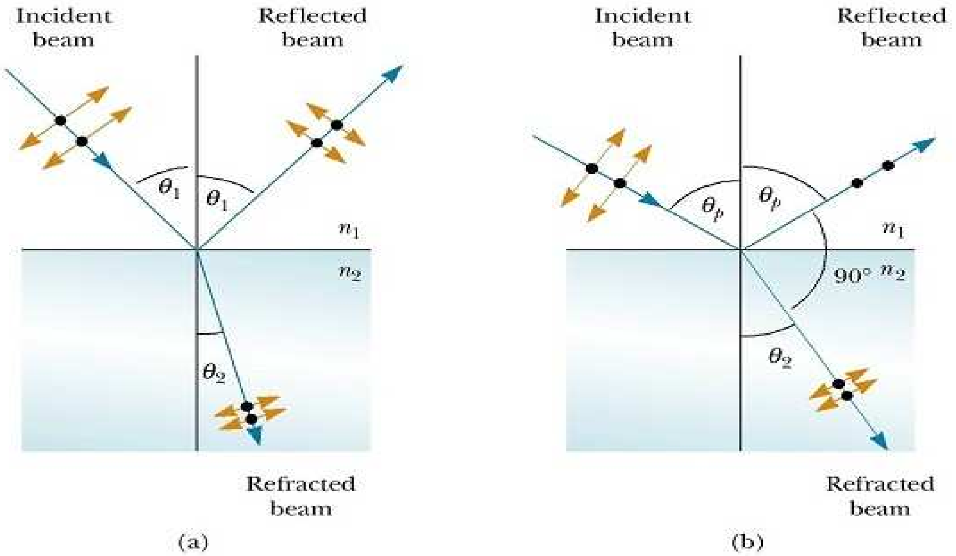
2. Both the coefficients (R & T) are only dependant on the angle of incidence θ_I and angle of refraction (transmission)θ_R (as per the both form of the equations).

Table 3.1: FRESNAL EQUATIONS

Co-efficients	**Transverse Electric**	**Transverse Magnetic**
R	$\frac{n_1\,cos\theta_I - n_2\,cos\theta_T}{n_1\,cos\theta_I + n_2\,cos\theta_T}$	$\frac{n_2\,cos\theta_I - n_1\,cos\theta_T}{n_2\,cos\theta_I + n_1\,cos\theta_T}$
T	$\frac{2\,n_1\,cos\theta_I}{n_1\,cos\theta_I + n_2\,cos\theta_T}$	$\frac{2\,n_1\,cos\theta_I}{n_1\,cos\theta_T + n_2\,cos\theta_I}$

3.5 Brewster's law

Brewsters law, relationship for light waves stating that the maximum polarization (vibration in one plane only) of a ray of light may be achieved by letting the ray fall on a surface of a transparent medium in such a way that the refracted ray makes an angle of 90^o with the reflected ray. The law is named after a Scottish physicist, Sir David Brewster, who first proposed it in 1811. To understand this let's look at the following picture. A ray of ordinary (nonpolarized) light of a given wavelength incident on a reflecting surface of a transparent medium (e.g., water or glass). Waves with the electric field component vibrating in the plane of the surface are indicated by short lines crossing the ray, and those vibrating at right angles to the surface, by dots. Most of the waves of the incident ray will be transmitted across the boundary (the surface of the water or glass) as a refracted ray making an angle r with the normal, the rest being reflected (part (a) of the picture). But for a specific angle of incidence (p), called the polarizing angle or Brewsters angle, the electric field component vibrating at right angles to the surface vanishes completely (part (b) of the picture).



3.5.1 Derivation of Brewster's law

Thus form the figure it is clear that at Brewster angle (ie the angle of incidence) the component of the electric field vibrating parallel to the plane of incidence doesn't reflect therefore the we can safly let the reflection co-efficient, R to be equal to zero. Starting with the Reflection coefficient for the parallel component of E field as

$$R = \frac{E_R}{E_T} = \frac{n_2\, cos\theta_I \,-\, n_1\, cos\theta_T}{n_2\, cos\theta_I \,+\, n_1\, cos\theta_T} = \frac{\frac{n_2}{n_1}\, cos\theta_I \,-\, cos\theta_T}{\frac{n_2}{n_1}\, cos\theta_I \,+\, cos\theta_T}$$

But $\dfrac{sin\theta_I}{sin\theta_T} = \dfrac{n_2}{n_1}$ and a furher simplification will lead to

$$\begin{aligned}
R &= \frac{sin\theta_I\, cos\theta_I \,-\, sin\theta_T\, cos\theta_T}{sin\theta_I\, cos\theta_I \,+\, sin\theta_T\, cos\theta_T} \\
&= \frac{tan\theta_I\, sec^2\theta_T \,-\, tan\theta_T\, sec^2\theta_I}{tan\theta_I\, sec^2\theta_I \,+\, tan\theta_T\, sec^2\theta_I} \qquad \text{Diving numerator and denominator by } (cos^2\theta_I\, cos^2\theta_T) \\
&= \frac{tan\theta_I\,(1 + tan^2\theta_T) \,-\, tan\theta_T\,(1 + tan^2\theta_I)}{tan\theta_I\,(1 + tan^2\theta_T) \,+\, tan\theta_T\,(1 + tan^2\theta_I)} \\
&= \frac{tan\theta_I \,+\, tan\theta_I\, tan^2\theta_T \,-\, tan\theta_T \,-\, tan\theta_T\, tan^2\theta_I}{tan\theta_I \,+\, tan\theta_I\, tan^2\theta_T \,+\, tan\theta_T \,+\, tan\theta_T\, tan^2\theta_I} \\
&= \frac{(tan\theta_I \,-\, tan\theta_T) \,-\, tan\theta_I\, tan\theta_T\,(tan\theta_I \,-\, tan\theta_T)}{(tan\theta_I \,+\, tan\theta_T) \,+\, tan\theta_I\, tan\theta_T\,(tan\theta_I \,+\, tan\theta_T)} \\
&= \frac{(tan\theta_I \,-\, tan\theta_T)\,(1 \,-\, tan\theta_I\, tan\theta_T)}{(tan\theta_I \,+\, tan\theta_T)\,(1 \,+\, tan\theta_I\, tan\theta_T)} = \frac{tan(\theta_I \,-\, \theta_T)}{tan(\theta_I \,+\, \theta_T)}
\end{aligned}$$

When R becomes equal to zero then the condition $\theta_I + \theta_T = \frac{\pi}{2}$ is satisfied and then the incident angle is called as Brwester's angle. Hence $\theta_T = \frac{\pi}{2} - \theta_B$. This will then lead to the Snell's law as the following

$$\begin{aligned}
\frac{sin\theta_I}{sin\theta_T} &= \frac{n_2}{n_1} \\
\frac{sin\theta_B}{sin(\frac{\pi}{2} - \theta_B)} &= \frac{n_2}{n_1} \\
\frac{sin\theta_B}{cos\theta_B} &= \frac{n_2}{n_1} \\
\boxed{tan\theta_B = \frac{n_2}{n_1}} & \qquad \text{Brewster's law}
\end{aligned}$$

3.6 Polarization of EM wave

Until now it seems that to describe light one must specify its frequency, its direction of propagation. But that's only two the third of complete description of the wave. In order to have the complete description one also must include and its state of polarization. It's must like a property of one-sidedness. The physical definition of wave polarization is actually the time behaviour of the electric field of a Transverse EM wave at a given point in space. In other words, the state of polarization of a wave is described by the geometrical shape which the tip of the electric field vector draws as a function of time at a given point in space. The plane in which the electric field oscillates is defined as the plane of polarization. Thus, polarization is a fundamental characteristic of a wave, and every wave has a definite state of polarization.
So let us assume that we have monochromatic light propagating along the +z direction. Light is a transverse electromagnetic wave, the electric field is always perpendicular to the direction of propagation. Because the direction of propagation is along the +z axis, the electric field vector, $\vec{E}$ must lie in the plane formed by the x and y axes. This can be expressed mathematically as follows:

$$\vec{E}\,(x,\,y,\,z,\,t) = \vec{E}_x(z,\,t)\hat{\imath}\,+\,\vec{E}_y(z,\,t)\hat{\jmath}$$

The components of the electric field E_x and E_y do not depend on x and y because we assume that the wave is a plane wave propagating along the +z direction. Let us now consider two waves with their electric fields oriented in and directions respectively.

For the x axis oriented wave

$$\vec{E}_x\;=\;\vec{E}_{0x}\,sin(\omega\,t\;-\;\kappa\,z)$$

For the y axis oriented wave

$$\vec{E}_y\;=\;\vec{E}_{0y}\,sin(\omega\,t\;-\;\kappa\,z\;+\delta)$$

where δ is the phase constant between the two waves and the amplitudes E_{0x} and E_{0y} are real constants. .

3.6.1 Linnear Polarization:

Suppose both $\vec{E}_x$ and $\vec{E}_y$ components which are in phase ie $\delta = 0$ having different magnitudes. The magnitudes of $\vec{E}_x$ and $\vec{E}_y$ reach their maximum and minimum values simultaneously as $\vec{E}_x$ and $\vec{E}_y$ are in phase. Without losing generality let us take $z = 0$ which will give

$$\vec{E}_x\;=\;\vec{E}_{0x}\,sin(\omega\,t\;-\;\kappa\,z)\qquad .....(1)$$

$$\vec{E}_y\;=\;\vec{E}_{0y}\,sin(\omega\,t\;-\;\kappa\,z)\qquad .....(2)$$

Now (1) divided by (2) will give

$$\frac{\vec{E}_x}{\vec{E}_y}\;=\;\frac{\vec{E}_{0x}}{\vec{E}_{0y}}$$

$$\vec{E}_x\;=\;\frac{\vec{E}_{0x}}{\vec{E}_{0y}}\,\vec{E}_y$$

So at any point on the positive z axis, the ratio of magnitudes both the components is constant. This is the equation of a straight line with slope $\frac{\vec{E}_{0x}}{\vec{E}_{0y}}$. The tip of electric field vector therefore draws a straight line if $\delta = 0$, irrespective of the amplitudes of the two field components. This polarization is hence called the **Linear Polarization** or the wave is said to be linearly polarized.

3.6.2 Circular Polarization:

If the two planes $\vec{E}_x$ and $\vec{E}_y$ (which are orthogonally polarized) are of equal in amplitude ($\vec{E}_{0x} = \vec{E}_{0y} = \vec{E}_0$) but has 90^o phase difference between them, then the resulting wave is circularly polarized. In such case at any instant of time, if the amplitude of the any one component is maximum, then other component amplitude becomes zero due to the phase difference. Thus the magnitude of the resultant vector $\vec{E}$ is constant at any instant of time, but the direction is the function of angle between the relative amplitudes of $\vec{E}_x$ and $\vec{E}_y$ at any instant.

$$\begin{aligned}\vec{E}_x\;&=\;\vec{E}_{0x}\,sin(\omega\,t\;-\;\kappa\,z)\\&=\;\vec{E}_0\,sin(\omega\,t\;-\;\kappa\,z)\qquad .....(1)\end{aligned}$$

$$\begin{aligned}\vec{E}_y\;&=\;\vec{E}_{0y}\,sin(\omega\,t\;-\;\kappa\,z\;+\;\frac{\pi}{2})\\&=\;\vec{E}_0\,cos(\omega\,t\;-\;\kappa\,z)\qquad .....(2)\end{aligned}$$

Now squaring and adding the last two equations we get

$$\vec{E}_x^2\;+\;\vec{E}_y^2\;=\;\vec{E}_0^2$$

If the resultant electric field $\vec{E}_x$ is projected on a plane perpendicular to the direction of propagation, then the locus of all such points is a circle (since the last equation is the equation of a circle centered at origin with radius $\vec{E}_0$) with the center on the z- axis. The circular polarization can be divided into Left-circular and Right-circular polarization.

• **Right circularly polarized:**
If $\delta = \frac{-\pi}{2}$ then vector $\vec{E}$ rotates clockwise ie to our right hand, then it is called the **RIGHT HANDED ROTATION** and light is called right circularly polarized.

• **Left circularly polarized:**
If $\delta = \frac{\pi}{2}$ then vector $\vec{E}$ rotates anticlockwise ie to our left hand, then it is called the **LEFT HANDED ROTATION** and light is called left circularly polarized.

3.6.3 Elliptical Polarization:

In most of the cases, the components of the wave have different amplitudes and are at different phase angles other than 90 degrees. This results the elliptical polarization. Consider that electric field has both components $\vec{E}_x$ and $\vec{E}_y$ which are not equal in amplitude and are not in phase. As the wave propagates, the maximum and minimum amplitude values of $\vec{E}_x$ and $\vec{E}_y$ not simultaneous and are occurring at different instants of the time. Thus the direction of resultant field vector varies with time.

$$\begin{aligned} \vec{E}_x &= \vec{E}_{0x}\, sin(\omega\, t - \kappa\, z) \\ &= \vec{E}_{0x}\, sin\, \omega\, t \qquad \text{at z=0} \\ \frac{\vec{E}_x}{\vec{E}_{0x}} &= sin\, \omega\, t \qquad .....(1) \end{aligned}$$

$$\begin{aligned} \vec{E}_y &= \vec{E}_{0y}\, sin(\omega\, t - \kappa\, z + \delta) \\ &= \vec{E}_{0y}\, sin(\omega\, t + \delta) \qquad \text{at z=0} \\ \frac{\vec{E}_y}{\vec{E}_{0y}} &= sin\,(\omega\, t + \delta) \\ &= sin\, \omega\, t\, cos\, \delta + cos\, \omega\, t\, sin\, \delta \qquad .....(2) \end{aligned}$$

Now eliminating $\omega\, t$ from equation (2) we get

$$\begin{aligned} \frac{\vec{E}_y}{\vec{E}_{0y}} &= \frac{\vec{E}_x}{\vec{E}_{0x}}\, cos\, \delta + \sqrt{1 - \frac{\vec{E}_x^2}{\vec{E}_{0x}^2}}\, sin\, \delta \\ \frac{\vec{E}_y}{\vec{E}_{0y}} - \frac{\vec{E}_x}{\vec{E}_{0x}}\, cos\, \delta &= \sqrt{1 - \frac{\vec{E}_x^2}{\vec{E}_{0x}^2}}\, sin\, \delta \\ \frac{\vec{E}_y^2}{\vec{E}_{0y}^2} - \frac{2\, \vec{E}_x\, \vec{E}_y\, cos\, \delta}{\vec{E}_{0x}\, \vec{E}_{0y}} + \frac{\vec{E}_x^2}{\vec{E}_{0x}^2}\, cos^2\, \delta &= \left(1 - \frac{\vec{E}_x^2}{\vec{E}_{0x}^2}\right) sin^2\, \delta \qquad \text{by squaring} \\ \frac{\vec{E}_y^2}{\vec{E}_{0y}^2} - \frac{2\, \vec{E}_x\, \vec{E}_y\, cos\, \delta}{\vec{E}_{0x}\, \vec{E}_{0y}} + \frac{\vec{E}_x^2}{\vec{E}_{0x}^2}\, cos^2\, \delta &= sin^2\, \delta \end{aligned}$$

If the locus of the end points of the field vector $\vec{E}$ traced then one can observe that the $\vec{E}$ moves elliptically on the plane. Hence such wave is called as elliptically polarized.